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Developing Mathematical Modelers

by Rose Mary Zbiek, Ph.D.

Rose Mary Zbiek, Author

Rose Mary Zbiek's scholarly work has focused on mathematical reasoning and representations in technology intensive environments. Dr. Zbiek also focuses on ensuring teachers have the support they need to implement math modeling tasks in the classroom.

Introduction

Mathematical modeling moved into the mainstream for many teachers, curriculum developers, teacher educators, and others in the United States largely due to its inclusion as both a high school conceptual category and a mathematical practice. Despite the attention, mathematical modeling is not yet clearly understood or accurately presented in many mathematics classrooms. Ensuring that all involved in mathematics instruction have a clear understanding of what mathematical modeling is and where it fits in high school mathematics is warranted.

The essence of mathematical modeling is the pursuit of authentic questions that originate outside mathematics. To understand mathematical modeling means to know it as a process and to distinguish between mathematical modeling problems and application problems as well as between mathematical modeling and modeling with mathematics.

Mathematical Modeling Defined and Described

Model and modeling are common terms used in different ways in everyday language and in mathematics. In mathematics teaching and learning, definitions—or at least descriptions—of mathematical modeling abound and appear in multiple venues. In terms of standards, "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (p. 7). Mathematicians whose work is mathematical modeling, in particular, the authors of *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)* (Garfunkel et al., 2016), describe mathematical modeling as "a process that uses mathematics to represent, analyze, and make predictions or otherwise provide insight into real-world phenomena" (p. 10).

Perhaps because mathematical modeling is commonly referred to as simply "modeling," it is often confused with other forms of "modeling" in the STEM fields, such as statistical modeling or engineering design. Peck, Gould, and Miller (2013) describe the essence of how mathematical models and statistical models differ:

> Bivariate, or two-variable, populations require functions of the form $y = f(x)$ to model the structure, which is the overall trend or relationship between two quantitative variables (p.15).

Whereas

> Statistical models extend the mathematical model by including a variability component. Statistical measures of variability, such as the standard deviation, give an indication of how much, on average, data values deviate from the structural part of the model (p.17).

Both statistical and mathematical modeling involve variables. However, attention to variance in data and ways to represent variance as part of a model is a key difference between mathematical modeling and statistical work.

Some scholars argue that a mathematical model should be defined more broadly as a system of elements that captures what matters about a familiar system for the purposes of describing, explaining, or predicting the familiar system (Doerr & English, 2003). Mathematical modeling so defined seems reminiscent of engineering design, which now is an essential feature of the curriculum in many schools due to its inclusion in Next Generation Science Standards (National Science Teachers Association, 2012). Engineering design differs from mathematical modeling as defined above in that the former represents systems or parts of systems. Development and evaluation of a design can be based in non-mathematical thinking and reasoning, including scientific theory. Interestingly, despite differences among work in statistics, engineering work, and mathematical modeling work, statistical work can be part of what is used in mathematical modeling and mathematical modeling can inform engineering design.

Mathematical modeling also must be distinguished from modeling with mathematics. As Cirillo,Pelesko, Felton-Koestler, and Rubel (2016) observe, “modeling mathematics refers to using representations of mathematics to communicate mathematical concepts or ideas” (p. 4). Actions such as using algebra tiles to illustrate completing the square or folding paper to create a parabola, or using straws and marshmallows to build physical representations of a polyhedron are examples of modeling with mathematics, and not instances of mathematical modeling. Modeling with mathematics begins and ends in mathematics; mathematical modeling begins and ends in the real world. Problems without a real-world context (see Figure 1) and problems in which the context remains completely in the background during the entire solution process (see Figure 2) are not mathematical modeling problems—they serve other purposes in high school mathematics. Mere inclusion of a real-world context in a problem does not make it a mathematical modeling problem—the problem might be an application task. To be a modeling task requires the context to be essentially inseparable from the mathematics.

> “The essence of mathematical modeling is the pursuit of authentic questions that originate outside mathematics.”

21. What is the equation of an ellipse with foci at (−8, 0) and (8, 0) that passes through the points (0, −3) and (0, 3)? SEE EXAMPLE 3

FIGURE 1: NO REAL WORLD CONTEXT

26. Oscar participates in a charity walk. The graph shows his distance in miles from the water stop as a function of time. How many miles did Oscar walk? Explain your answer. SEE EXAMPLE 3

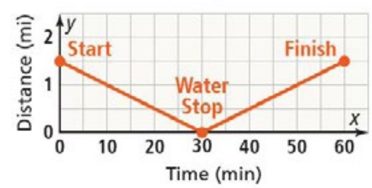


FIGURE 2: Real-world context essentially irrelevant

Arguably, the vast majority of problems that appear in most school mathematics materials are applications problems, not mathematical modeling tasks. Henry Pollak, renowned mathematician and modeler, succinctly distinguished mathematical modeling from applications of mathematics, noting that mathematical modeling includes: “(1) explicit attention at the beginning of the process of getting from the problem outside of mathematics to its mathematical formulation, and (2) an explicit reconciliation between the mathematics and the real-world situation at the end” (2003, p. 649).

Mathematical Modeling as a Process

Pollak’s allusion to the beginning and the end of the work involved, positions mathematical modeling as a process, consistent with the aforementioned definitions of mathematical modeling. The process is inherently

> Mathematical modeling is not a procedure or algorithm. There is no fixed or firm set of steps that take one from the beginning to the end of a mathematical modeling event.

iterative—but not in the sense of simple recursion. Mathematical modeling is not a procedure or algorithm. There is no fixed or firm set of steps that take one from the beginning to the end of a mathematical modeling event. The stops, restarts, and do-overs of mathematical modeling often come from the need to "reflect on whether the results make sense, possibly improving the model if it has not served its purpose" (NGA Center & CCSSO, 2010, p. 7). Revisions can be many, and the process might need to be called to a close when a useful, though not perfect, model is achieved.

Many diagrams have been offered as ways to capture mathematical modeling as an iterative process grounded in the real world. They may differ in the number of nodes or connections they include or in how these elements are labeled, but they are similar in that they involve probing a real-world situation, carrying out some mathematical work, and validating a model.

Because most of school mathematics has long focused almost exclusively on learning and doing the almost purely mathematical work, a diagram that articulates aspects of mathematical modeling can prompt images of the critical role that the real-world contexts play in mathematical modeling—and the centrality of the real-world context is key to what makes mathematical modeling different from applied mathematical work and typical mathematical problem solving. Bliss, Fowler, and Galluzzo (2014) provide such a diagram (see Figure 3).

Like most modeling diagrams, it shows that the mathematical modeling process begins with a real-world problem, ends with reporting results, and provides a definite space for analysis and model assessment. Most importantly, the diagram highlights the three research and brainstorming activities of defining the problem, defining the variables, and making assumptions. The arrows emphasize the back-and-forth movement among these elements. This aspect of the diagram directly hits the goal for students to be "comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later" (NGA Center & CCSSO, 2010, p.7). Getting a solution—the mathematical work that does not distinguish mathematical modeling from other mathematical work—is a very small part of the overall process.

Mathematical Modeling and School Mathematics

The process of mathematical modeling is time-consuming and messy. When encountering a mathematical modeling problem and entering into the activity of research and brainstorming, a modeler (or teacher) might not know how long the work will take or what mathematics will be needed. The multiple valid solutions, dead ends, and revisions are not in sync with a popular conception of mathematics as a well defined rapid run from a problem to its single correct solution that has dominated mathematics for decades (Schoenfeld 1988). Different sets of assumptions can lead to drastically different—yet similarly valid—models and solutions. The differences are not merely different strategies for solving problems, such as using substitution, elimination, and graphing as different strategies for solving a system of linear equations. Different solutions and different models arise from choosing among related mathematical objects, such as different types of functions. The differences can be drastic. For example, different assumptions might lead to different linear equations, more equations, non-linear equations, equalities, or even new or different variables.

The non-predetermined nature of mathematical modeling contributes to the messiness of mathematical modeling that lies in contrast to the organization of mathematical ideas in the school curriculum. Multiple valid solutions, in addition to multiple strategies, compete with common expectations

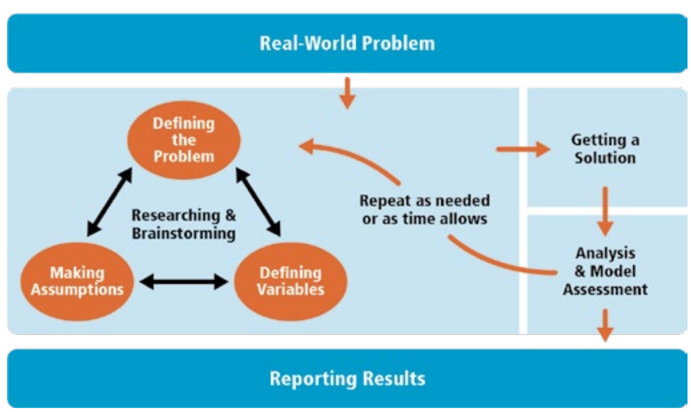


FIGURE 3: Diagram illustrating mathematical modeling as a process, adapted from Math Modeling: Getting Started & Getting Solutions https://m3challenge.siam.org/resources/modeling-handbook (Bliss, Fowler, & Galluzzo, 2014, p.6)

of what many people have long believed good mathematics should be: quickly solved problems with single correct answers (Schoenfeld, 1988). Mathematical modeling and school mathematics curricula also have competing goals in that the former is open to whatever mathematics is needed while the latter typically has a clearly articulated set of topics and expectations.

Despite these competing goals, student engagement in mathematical modeling has the potential to help students develop an understanding of curricula mathematics—the mathematics to be learned in the classroom at the students' current point in the curriculum (Zbiek & Conner, 2006).

Teachers and other educators can benefit from more productive beliefs about the doing, learning, and teaching of mathematical modeling (Zbiek, 2016a) as they enact curriculum materials that take a distributed atomistic approach to mathematical modeling.

Transitioning to Mathematical Modeling Work

Ideally, high school students should engage in big, messy mathematical modeling problems, but implementing a curriculum that fully embraces mathematical modeling can be challenging. Geiger, Ärlebäck, and Frejd (2016) contrast a holistic approach with an atomistic approach of mathematical modeling in the classroom. With a holistic approach, students engage in the complete mathematical modeling process; within an atomistic approach, students attend to particular aspects of the modeling process at different times. An atomistic approach seems productive when available time and student familiarity with mathematical modeling are issues.

Teachers may want to consider a distributed atomistic approach that involves opportunities to make various approaches explicit as they arise (Zbiek, 2016b). In a distributed atomistic approach, students engage frequently with particular components of the modeling process over time, with the goal of helping them become competent modelers able to engage in full-fledged mathematical modeling work. Incorporating such an approach into high school classrooms requires careful consideration of the types of problems students are asked to solve and what students are expected to do.

A transition from a typical high school mathematics experience to one that fully embraces mathematical modeling work should start with abandoning commonly held expectations for mathematics learning. For example, a focus on procedures, a common mainstay of the highschool mathematics classroom, is a problem not only in terms of the absence of math thinking and reasoning, but also in the way in which procedures are often taught using worked examples. While such examples can be powerful tools for learners (Star et al., 2015), they could also be barriers to mathematical modeling work if teachers and students anticipate worked examples (Davis, 2009).

One step in building mathematical modelers is explicit work analyzing models of others or from others' attempts to provide models. Students may be asked to use a function model and evaluate the output for a given input value (see example in Figure 4a), to analyze and correct or solve an equation or inequality—or system of equations and/or inequalities, given an input or output value (see example in Figure 4b). Or students may be asked to calculate and measure and observe or verify relationships based on prescribed geometry figures superimposed on a photo or sketches of a real-world object (see example in Figure 4c). These tasks are seemingly sufficient in meeting the expectation that students "can analyze those relationships mathematically to draw conclusions" (NGA Center & CCSSO, 2010, p.7).

35. Apply Math Models A teacher adjusts the grades of an exam using a curve. If a student's raw score on a test is x, the score based on the curve is given by the function $c(x) = 10\sqrt{x}$.
Five students received raw scores of 49, 42, 55, and 72. What are their scores according to the curve?

FIGURE 4a: Example of tasks for which students work with others' models

11. Error Analysis Describe and correct the error a student made when interpreting the graph of the vertical motion model $h(t) = -at^2 + bt + c$.

FIGURE 4b: Example of tasks for which students work with and identify errors in others' models

EXAMPLE 4 Use a Parabola to Model a Real-World Situation

One type of solar cooker is a reflective parabolic dish. Its cross section is a parabola modeled by the equation $y = \frac{1}{108}x^2$, with distances measured in inches. The bracket that holds the pan for the cooker needs to be placed at the focus. Assuming that the vertex of the parabolic dish is at the point $(0, 0)$, at what coordinates should the bracket be placed?

SOLUTION

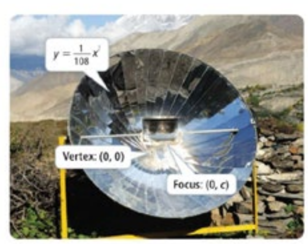


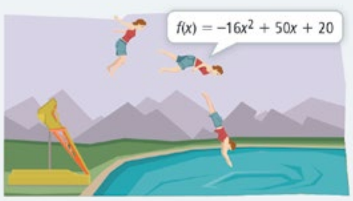
FIGURE 4c: Example of tasks for which students work with others' models

In a distributed atomistic approach, students should also be asked to modify or evaluate models given. Davis (2009) captures one aspect of the emphasis on real-world problems in a way that supports modeler development. He distinguishes between Full Domain (FD) graphs of functions as those that convey all key features of a function and Limited Domain (LD) graphs—graphs of otherwise common functions with domains that relate to a realistic context. Prospective modelers can be asked about the implications when LD graphs are used (see Figure 5). The ability to work with LD graphs is necessary to achieve the high school modeling goal of students being “able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas” (p.7).

Building on the atomistic approach of bringing modeling into daily lessons, teachers could provide opportunities for students to develop models that take several forms, such as representing patterns—or terms of a pattern—with linear functions or arithmetic sequences (see Figure 6), using particular algebraic forms to determine parameters to match particular settings (see Figure 7). All of these tasks prescribe the type of model, engage students in making few decisions, and often end with “correct” models.

Interestingly, Gould’s (2016) study of a few Common Core-aligned textbooks suggest that the modeling activities to which students are exposed are mainly analyze and interpret tasks—tasks that fit mainly in the Getting a Solution part of Bliss, Fowler, and Galluzzo’s diagram (see Figure 3). She argues that students need more experience determining variables and assumptions; they need to identify tasks that target the Research and Brainstorm components. The task in Figure 8 is an open-ended problem that requires students to attend to real-world problems.

39. Performance Task A human catapult is used to launch a person into a lake. This launch is modeled as shown, where x is the time in seconds from the launch.



PART A What equation can you use to find when the person touches the lake? Solve the equation and find the solution.

PART B Are your solutions the same for the equation and problem? Why or why not?

PART C What is the greatest height reached?

FIGURE 5: Example of reflection on Limited Domain graph for mode

35. Analyze and Persevere On October 1, Nadia starts a push-up challenge by doing 18 push-ups. On October 2, she does 21 push-ups. On October 3, she does 24 push-ups. She continues until October 16, when she does the final push-ups in the challenge.

a. Write an explicit definition to model the number of push-ups Nadia does each day.

b. Write a recursive definition to model the number of push-ups Nadia does each day.

c. How many push-ups will Nadia do on October 16?

d. What is the total number of push-ups Nadia does from October 1 to October 16?

FIGURE 6: Example of expressing a model based on an identified pattern

38. Apply Math Models The table shows the winning times for the 100-meter run in the Olympics since 1928. What is the equation of the line of best fit for the data? What do the slope and y-intercept represent? Estimate the winning time in 2010 and predict the winning time in 2020.

Year	Time (s)	Year	Time (s)
1928	10.80	1980	10.25
1932	10.30	1984	9.99
1936	10.30	1988	9.92
1948	10.30	1992	9.96
1952	10.40	1996	9.84
1956	10.50	2000	9.87
1960	10.20	2004	9.85
1964	10.00	2008	9.69
1968	9.95	2012	9.63
1972	10.14	2016	9.81
1976	10.06		

FIGURE 7: Example of expressing a model based on an identified pattern

27. Mathematical Connections Dakota bought 120 ft of wire fencing at \$0.50/ft to enclose a rectangular playground. The playground surface will be covered with mulch at a cost of \$1.25/ft^2. Write a quadratic function that can be used to determine the total cost of fencing and mulch for a playground with side length x. What is the cost if one side is 20 ft?

FIGURE 8: Example of expressing a model based on an identified pattern

Students also need validation tasks. The validation task in Figure 9 does not ask whether the model is the "right" model; rather students consider the aptness of the model by looking at how well it might fit an altered situation with different parameters and variables.

Mathematical modeling as a process includes revisiting initial solutions and reiterating components of the cycle. The problems that launch lessons offer opportunities for students to revisit the problem with altered situations or assumptions. These alterations should lead students to realize the need for new mathematical tools. Students explore finding a location equidistant from three points on a map, creating the need for finding the point that is equidistant from the vertices of a triangle (see Figure 10). The need to organize and then manipulate data related to both size and color creates a need for a matrix rather than a table (see Figure 11).

The atomistic approach focuses students on discrete components of the mathematical modeling process. As they develop modeling skills, emerging modelers need to orchestrate the elements into one or more mathematical modeling cycles. The *Mathematical Modeling in 3-Acts* tasks found in each topic of *enVision A|G|A* are one way to provide this bridge. These tasks are not full-fledged modeling activities—they generally have an intended solution path and a singular correct answer. Still, these tasks do fit the atomistic view in that they allow students to put together elements of the modeling process on their path to becoming proficient modelers. Judicious use of these tasks is consistent with the goal for students to "routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose." (NGA Center & CCSSO, 2010).

Students move into more authentic mathematical modeling experiences as they engage in components of the mathematical modeling process beyond mathematical manipulations and ideas—the Getting a Solution part in Figure 3. Important in the success of the atomistic approach is the extent to which students orchestrate components and make decisions. Routinely infusing curricular mathematics problems with modeling components singularly and in combination allows students to decide such things as which variables to use and what assumptions to make, which mathematical techniques to use, how the model will be analyzed, and what will be reported.

Also important are the opportunities for students to engage in full modeling activities.

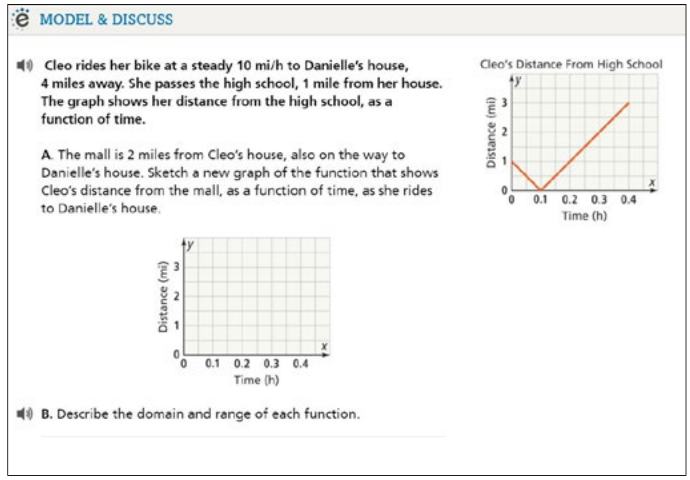


FIGURE 9: Example of expressing a model based on an identified pattern

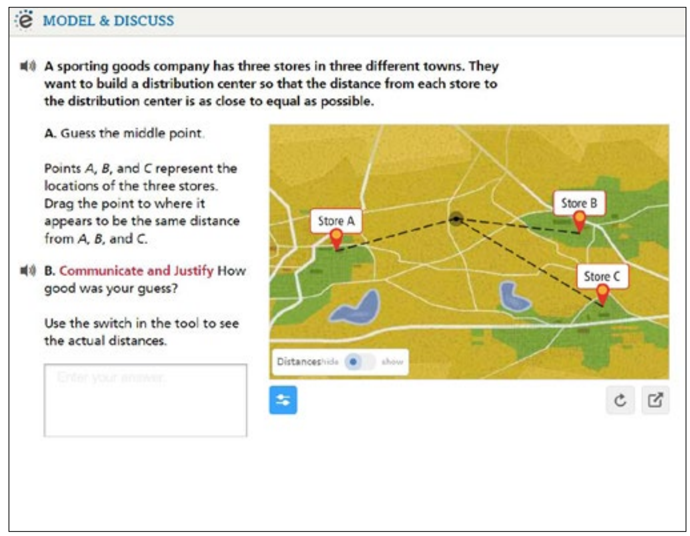


FIGURE 10: Lesson Launch that introduces the need for finding the point that is equidistant from the vertices of a triangle.

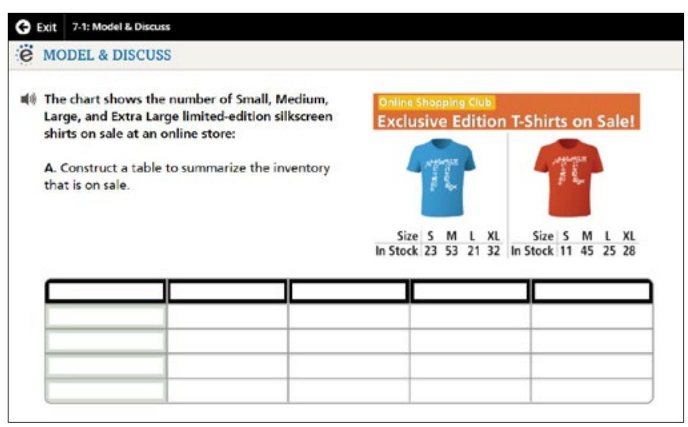


FIGURE 11: Lesson launch that introduce the need for matrix as a new mathematical tool

Summary

A distributed atomistic approach incorporates mathematical modeling into school mathematics in a way that not only develops students' modeling capacity but also supports students' learning of curricular mathematics as they focus on components of the modeling process. Students can experience modeling moments within daily *enVision A|G|A* lessons with problems that are infused with modeling components. This approach makes mathematical modeling a familiar mathematical practice and supports student learning of high school mathematics content.

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