Research
into Practice:
MATHEMATICS

Developing Mathematical Thinking Habits Through Mathematical Practices: Kindergarten Through Grade 5

BY DR. JANE SCHIELACK

In a world of technology with lightning-fast processing and vast memory capacity, what mathematical abilities are going to prepare students to be educated, productive, and employable members of society? The answer is that students must be prepared not only with the knowledge of important mathematical concepts and skills, but also with experiences in which they learn how to think about and use that knowledge in order to investigate, evaluate, make decisions, and justify choices. Over the past 30 years, national and state efforts to guide mathematics curricula have encompassed this type of thinking in mathematical processes (NCTM, 2000), strands of mathematical proficiency (NRC, 2001), mathematical habits of mind (Goldenberg, Shteingold, & Feurzeig, 2003; Levasseur & Cuoco, 2003; NRC, 2005), classroom mathematical practices (Cobb et al., 2011), and Standards for Mathematical Practice (MPs) (Mateas, 2016) that now appear in the current mathematics curricula of many states. For purposes of example, the ***enVision***® *Mathematics* program will be used to demonstrate how current mathematics programs incorporate the Standards for Mathematical Practice. In addition, descriptions will be included of the expected mental activity from students in Kindergarten through Grade 5 as they develop important mathematical thinking habits while learning mathematical content.

Standards for Mathematical Practice and Related Mathematical Thinking Habits

Mathematical thinking habits develop over time as students consider questions that require them to analyze, justify, and revisit mathematical representations and problem-solving processes (Figure 1). Students' mathematical understanding is strengthened when their experiences with mathematical content and mathematical practices intersect. When mathematical thinking habits are firmly entrenched in students' minds during the elementary grades, they will be prepared for continued development of these habits with more sophisticated content throughout middle school and high school.

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Mathematical Practices	Questions for Developing Mathematical Thinking Habits [from **enVision**® Mathematics, (Charles et al., 2020).]
MP1: Make sense of problems and persevere in solving them.	• What do I need to find? • What do I know? • What's my plan for solving the problem? • What else can I try if I get stuck? • How can I check that my solution makes sense?
MP2: Reason abstractly and quantitatively.	• What do the numbers and symbols in the problem mean? • How are the numbers or quantities related? • How can I represent a word problem using pictures, numbers, or equations?
MP3: Construct viable arguments and critique the reasoning of others.	• How can I use numbers, objects, drawings, or actions to justify my argument? • Am I using numbers and symbols correctly? • Is my explanation clear and complete?
MP4: Model with mathematics.	• How can I use math I know to help solve this problem? • How can I use pictures, objects, or an equation to represent the problem? • How can I use numbers, words, and symbols to solve the problem?
MP5: Use appropriate tools strategically.	• Which tools can I use? • Why should I use this tool to help me solve the problem? • Is there a different tool I could use? • Am I using the tool appropriately?
MP6: Attend to precision.	• Am I using numbers, units, and symbols appropriately? • Am I using the correct definitions? • Am I calculating accurately? • Is my answer clear?
MP7: Look for and make use of structure.	• What patterns can I see and describe? • How can I use the patterns to solve the problem? • Can I see expressions and objects in different ways?
MP8: Look for and express regularity in repeated reasoning.	• Are any calculations repeated? • Can I generalize from examples? • What shortcuts do I notice?

Source: National Council of Teachers of Mathematics (NCTM). Principles and Standards for School Mathematics (2000).

Figure 1. Mathematical Practices and Related Mathematical Thinking Habits

Encouraging Mathematical Thinking Habits

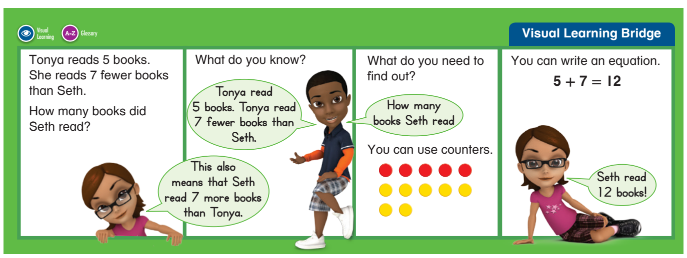
One of the best ways for students to internalize mathematical thinking habits is by seeing and hearing their teachers model the questions listed in Figure 1 so that they learn to ask those questions of themselves when presented with a problem to solve. When a student is involved in any problem-solving situation, whether in the classroom or outside of it, that student must learn to think automatically, “How can I use numbers in this situation? What patterns do I see and how can I use them? What is another way to represent this information, and what do I learn from it?” Students will build a sense of internal satisfaction when their thinking leads them to understand a new idea or make a good decision. The goal is for this type of mathematical thinking to become a habit, not something that someone else tells them to do! Therefore, students need to have opportunities to use the mathematical practices in situations where they can see the results of their thinking and receive and analyze feedback about their choices.

Mathematical Practices Across Kindergarten Through Grade 5

The coverage of each mathematical practice extends across grade levels as well as within each grade level. Although it is not expected (or reasonable) for a student to engage in *every* mathematical practice in *every* lesson, as students move through the elementary grades they have numerous and increasingly complex opportunities for experiences with each of the mathematical practices, building confidence in their ability, as well as willingness, to think mathematically. Looking at each mathematical practice individually across different grade levels helps teachers identify what mathematical thinking looks and sounds like in the classroom.

MP1: MAKE SENSE OF PROBLEMS AND PERSEVERE IN SOLVING THEM.

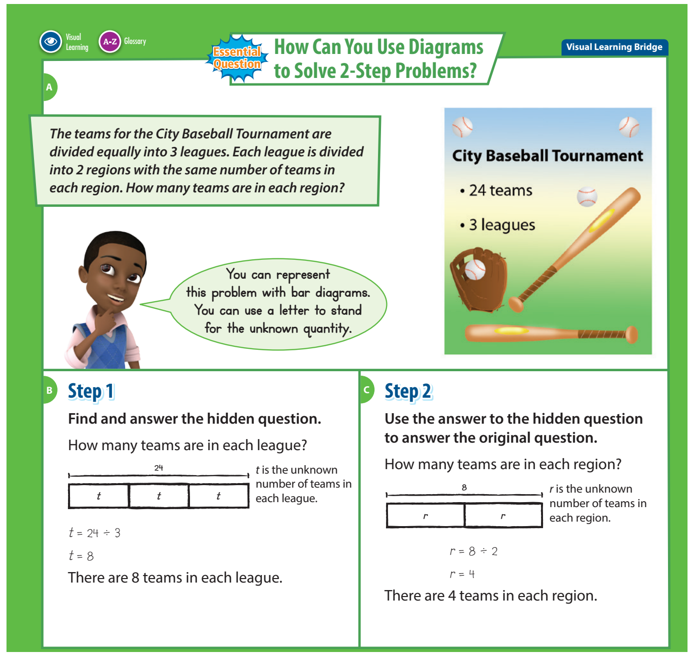
As students advance through the elementary grades, they develop a toolkit of strategies that they can use to help them "make sense of problems and persevere in solving them." For example, students in Kindergarten and Grade 1 are asked to demonstrate their understanding of a problem by acting it out—perhaps by joining and separating sets of counters or by sorting objects based on their attributes. Teachers help them to connect their language describing the mathematics in the situation—e.g., 5 objects and 7 objects joined together are equal to a total of 12 objects—to representations with equations such as 5 + 7 = 12.



Source: ***enVision***® *Mathematics ©2020, Grade 1, Lesson 3-8, p.138 (Charles et al., 2020).*

Figure 2. Grade 1 students use counters to act out a problem.

In Grades 2 and 3, students' toolkit of problem-solving strategies continues to grow. They focus on using strategies, such as drawings of bar diagrams, to help them make sense of problems. At the same time, students' verbal and written explanations are becoming more complex.



Source: ***enVision***® *Mathematics ©2020 Grade 3, Lesson 11-2, p. 414 (Charles et al., 2020).*

Figure 3. Grade 3 students draw bar diagrams to represent problems.

In Grades 4 and 5, students continue to demonstrate their understanding of a problem by restating it in their own words, recognizing the question that is being asked, and identifying and representing the information that is needed to answer a question that could involve combinations of operations, measurement, or geometry concepts.

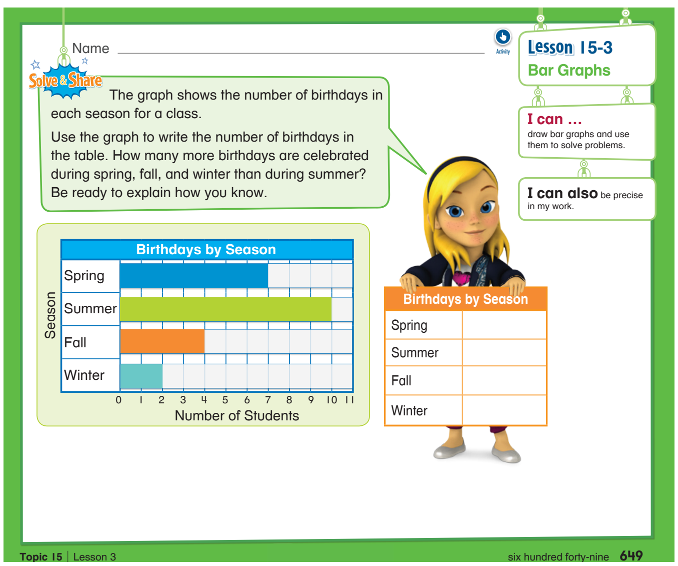
Willingness to persevere in solving a problem increases with confidence that grows from experience. One of the reasons students do not persevere in solving a particular problem is that they exhaust their ability to think about it. At all levels, students who are encouraged to share their thinking with others increase their ability to see problems from different perspectives. They draw on these shared examples as they work on future problems. Students use number sense and estimation to think about the reasonableness of a solution and determine whether or not they need to continue working on a certain problem.

MP2: REASON ABSTRACTLY AND QUANTITATIVELY.

When students focus their attention on either applying the meanings of numbers or operations to a given situation or deriving meanings from the numbers or operations involved in a given situation, they are thinking abstractly and quantitatively.

At Kindergarten and Grade 1, it is beneficial for teachers to model how to use quantities to describe situations and then have students do the same. As students gain confidence in using number names and vocabulary for appropriate operations, they use these ideas in a variety of situations, such as giving directions as they work at the water table in Kindergarten or writing about science concepts in Grade 1.

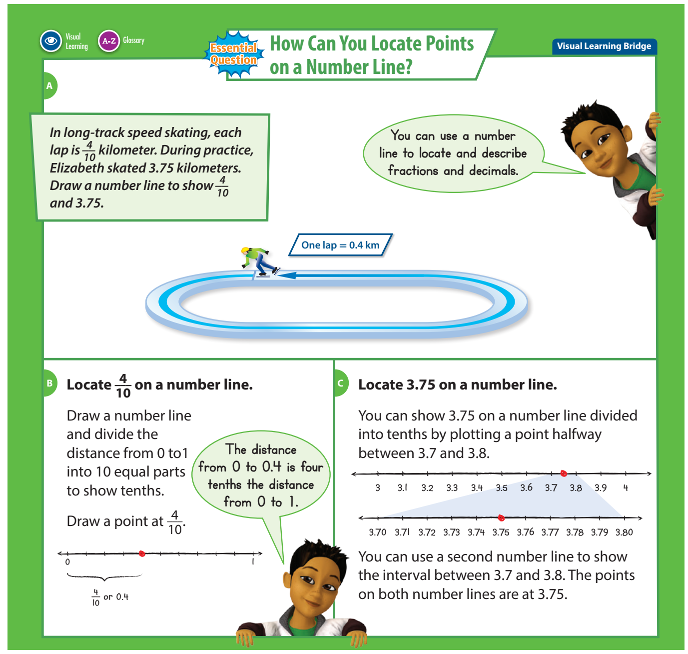
In Grades 2 and 3, students continue to think quantitatively by applying questions about how many, how many more, how many less, and how many total in a variety of contexts. They use formal quantitative representations such as bar graphs to facilitate their numerical reasoning.



*Source: **enVision**® Mathematics ©2020, Grade 2, Lesson 15-3, p. 649 (Charles et al., 2020).*

Figure 4. Grade 2 students use bar graphs and numerical reasoning to solve problems.

As students work with multiplication, division, and fractions in Grades 4 and 5, teachers expand questions from the previous grade levels to include questions that require fractional rather than whole number answers (e.g., How much of the area is shaded?) and questions that refer to multiplicative comparisons (e.g., How does the distance from 0 to 0.4 compare to the distance from 0 to 1 on a number line?)



*Source: **enVision**® Mathematics ©2020, Grade 4, Lesson 12-2, p. 450 (Charles et al., 2020).*

Figure 5. Grade 4 students use multiplicative reasoning to solve problems that involve fractions and decimals.

As they strengthen their quantitative thinking, students at all levels more often use numbers and operations in situations across different areas of the curriculum, and not just in mathematics class.

MP3: CONSTRUCT VIABLE ARGUMENTS AND CRITIQUE THE REASONING OF OTHERS.

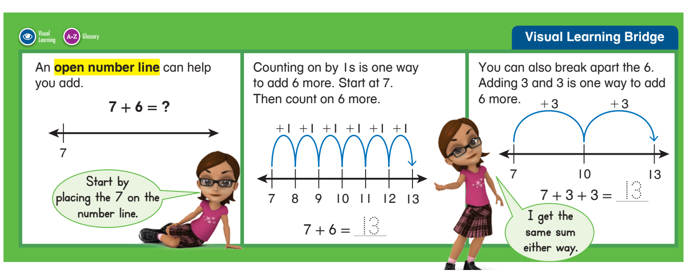
Students are engaged in mathematical thinking habits when they are encouraged to justify their own thinking about a mathematical situation and to evaluate the reasonableness of other students' thinking. A simple, but powerful, approach to accomplishing this goal is for the class to adopt the motto "Convince Me!" This expectation enables all students to provide valid explanations of their actions and the thinking behind them, as well as to examine other students' explanations. Development of this mathematical practice across the grades involves the maturation of communication skills along with the knowledge of mathematics content. At all grade levels, students use objects, pictures, words, and symbols in their explanations and justifications. Different grouping arrangements based on students' experiences with talking about mathematics lead to productive discourse.

Students in Kindergarten and Grade 1 might interact in pairs, or one-on-one with the teacher. Students in Grades 2 and 3 learn respectful methods of critique by working in small groups of 3 or 4, practicing their arguments within their group, and then sharing their explanation with another group. In Grades 4 and 5, students are given opportunities to share their thinking with the whole class, and members of the class are asked to provide critical comments in a respectful way. Although teachers often ask students to explain why they made certain decisions in order for the teacher to understand their thought processes and uncover misconceptions, students also share their thinking to convince others that they have made valid logical arguments. Embracing the motto "Convince Me!" guarantees that perspective.

MP4: MODEL WITH MATHEMATICS.

The phrase "model with mathematics" refers to representing a situation with appropriate numbers and operations, diagrams, tables, or graphs. These mathematical models are used to capture the essential characteristics of a problem in order to explore, analyze, and solve the problem. As students experience the use of mathematical models, they move from interpreting given models to creating them and being able to explain their own choices of models.

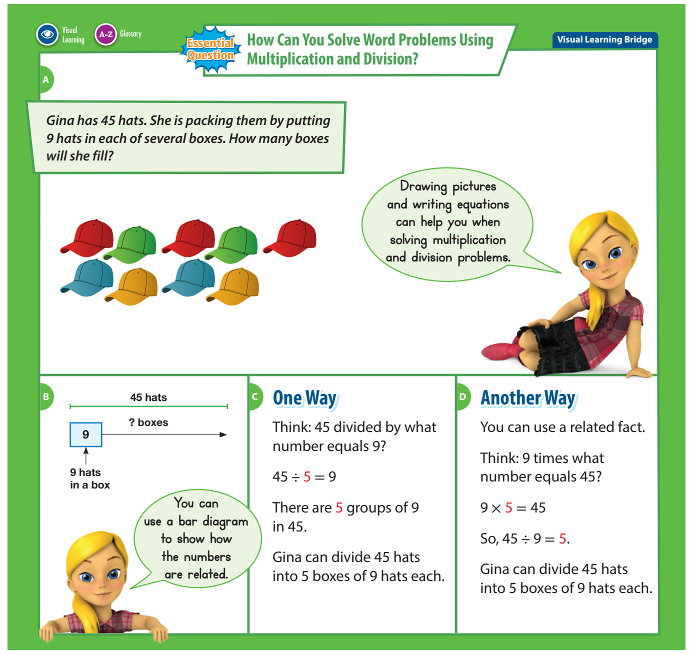
Students in Kindergarten and Grade 1 model with mathematics when they connect their action of joining 2 counters and 3 counters to make a set of 5 counters to the words, "2 and 3 is five," and then to the equation, 2 + 3 = 5. They learn to use the number line as a model for the whole-number system on which they can represent numbers and operations.



*Source: **enVision**® Mathematics ©2020, Grade 1, Lesson 3-2, p. 114 (Charles et al., 2020).*

Figure 6. Grade 1 students learn how to use an open number line to represent addition.

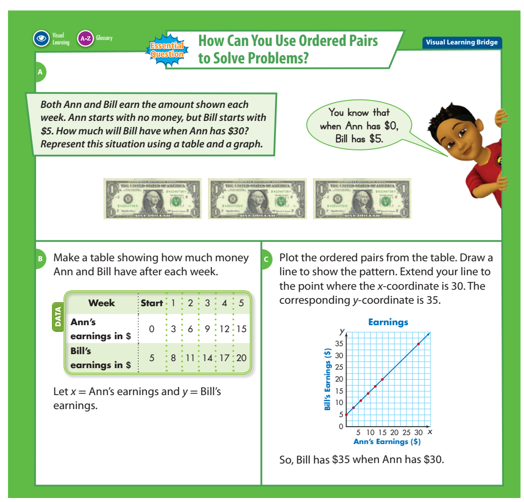
In Grades 2 and 3, students add more compact mathematical models, such as bar diagrams, to their repertoire.



*Source: **enVision**® Mathematics ©2020, Grade 3, Lesson 5-4, p. 182 (Charles et al., 2020).*

Figure 7. Grade 3 students draw mathematical models to represent multiplication and division.

Students in Grades 4 and 5 expand their use of number lines to include fractions and negative numbers and combine two number lines to create a coordinate graph that can be used to model positions on a plane.



*Source: **enVision**® Mathematics ©2020, Grade 5, Lesson 14-3, p. 574 (Charles et al., 2020).*

Figure 8. Grade 5 students graph ordered pairs to analyze and solve problems.

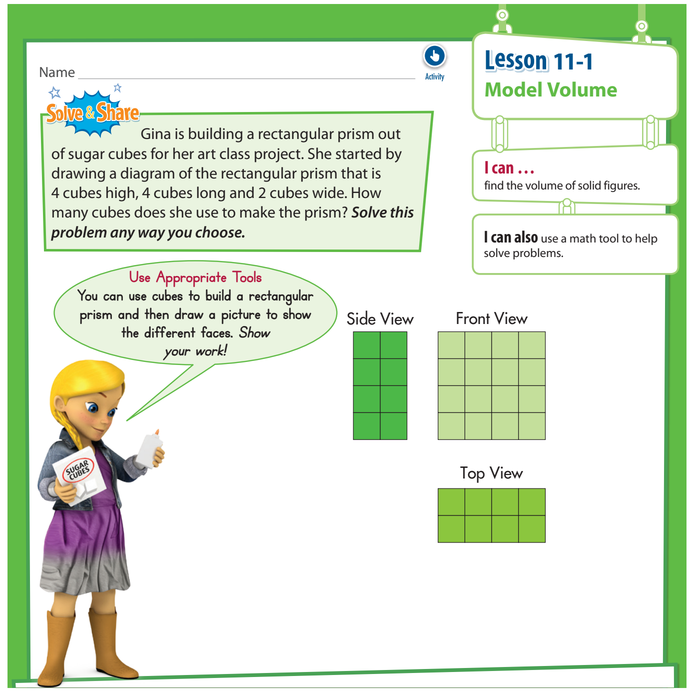
Across the grade levels, students work with increasingly complex charts and tables to search for and explain patterns that lead to valid conclusions.

MP5: USE APPROPRIATE TOOLS STRATEGICALLY.

Place-value blocks, linking cubes, paper and pencil, calculators, rulers, compasses, geometric solids, geoboards, spinners, and a multitude of other materials can be considered “tools of the trade” for teaching, learning, and doing mathematics. However, it is not enough for students to be able to use these tools; they must be able to select the best tool for solving a problem in a real-world or mathematical context. For example, a calculator is not an appropriate tool for a fourth-grader to use to compute the answer to 32 + 45; however, a student in Kindergarten or Grade 1 who has used objects to figure out that 3 and 1 is 4 and then enters the symbols, 3 + 1 =, into a calculator to create the expected result of 4, is learning how to “talk” to the calculator so that he or she can eventually use, and evaluate the use of, technology in more difficult situations. It is important for teachers to introduce the tools in appropriate situations, highlight their usefulness, and provide a “toolbox” for exploring and doing mathematics.

In Kindergarten and Grade 1, after a teacher uses counters and connecting cubes to model patterns for “one more” and “one less” counters and connecting cubes become an appropriate tool for students to choose for solving other problems involving counting. In Grades 2 and 3, students are given experiences with a variety of place-value tools and then choose from those tools as they build facility with new algorithms.

If the tools they encountered in earlier grades are flexible in their use, students in Grades 4 and 5 will be able to apply familiar tools in new settings, such as using connecting cubes to explore formulas for area and volume.



Source: ***enVision***® *Mathematics ©2020, Grade 5, Lesson 11-1, p. 457 (Charles et al., 2020).*

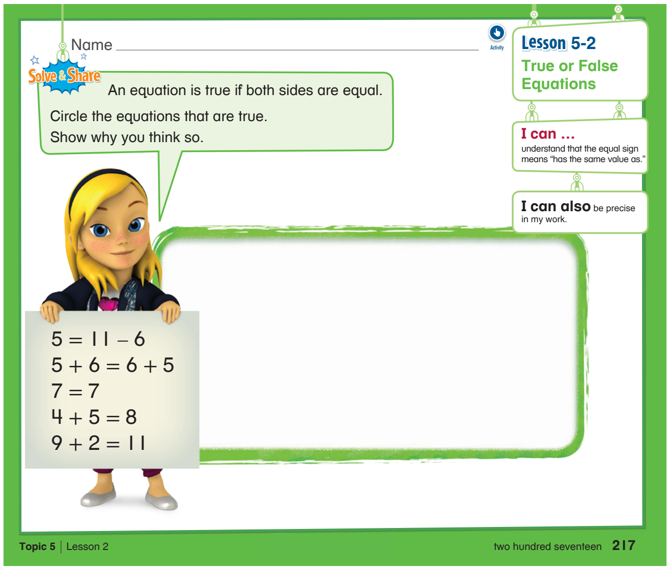
Figure 9. Grade 5 students choose and use tools to make connections between 2-D drawings and 3-D objects.

Every possible tool has the potential for misuse or effective use, and it is important that teachers help students learn to select the tool(s) that make the most sense for the task at hand. As students use different tools, discussions about why one tool is a better choice than another in a given situation build students’ expertise in selecting various tools and in evaluating their appropriateness.

MP6: ATTEND TO PRECISION.

Although it is important for students to learn to calculate accurately and efficiently, precision in this mathematical practice refers to more than precision in computation. It also refers to students’ use of language in solving problems and in explaining what they have done.

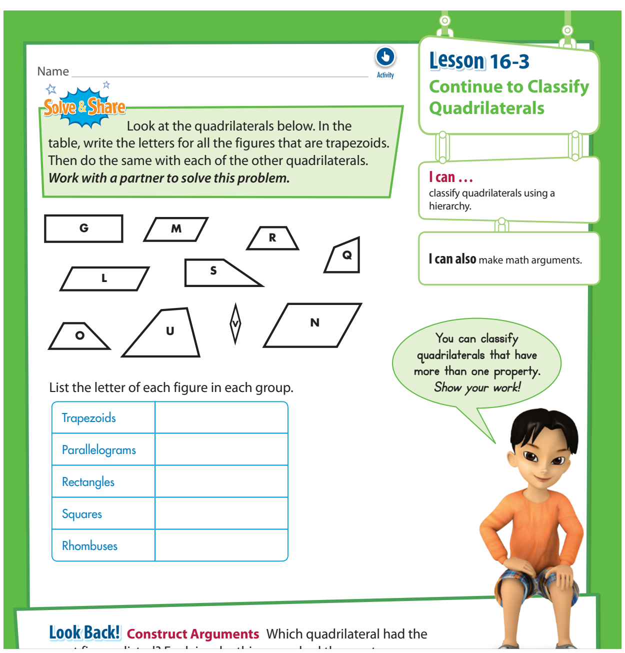
At Kindergarten and Grade 1, teachers need to be precise with their own language as they help students distinguish between related categories such as “true” and “not true.”



*Source: **enVision**® Mathematics ©2020, Grade 1, Lesson 5-2, p. 217 (Charles et al., 2020).*

Figure 10. Grade 1 students apply their understanding of true and false to equations.

In Grades 2 and 3, students build their mathematical vocabulary and its proper use by listening to descriptions of mathematical thinking as they work with partners and by providing feedback that encourages precision in these discussions. In Grades 4 and 5, students strengthen their use of mathematical language as they learn and use definitions that clearly delineate the similarities and differences among related concepts such as squares and rectangles.



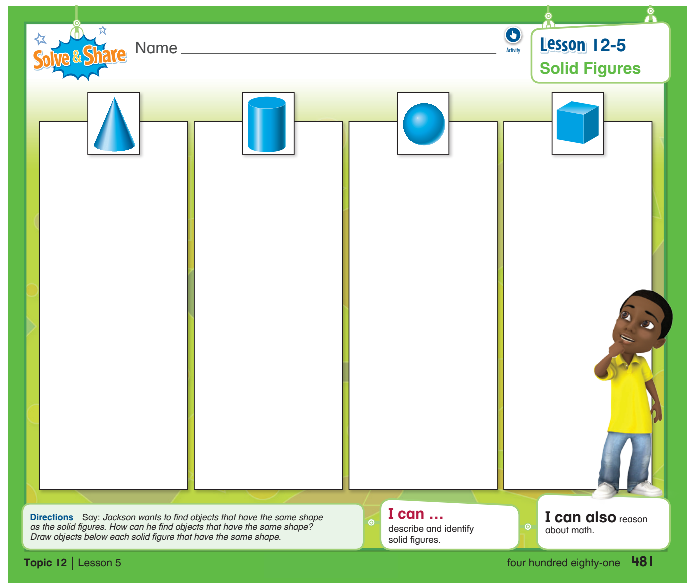
*Source: **enVision**® Mathematics ©2020, Grade 5, Lesson 16-3, p. 629 (Charles et al., 2020).*

Figure 11. Grade 5 students use precise language and definitions to classify quadrilaterals.

MP7: LOOK FOR AND MAKE USE OF STRUCTURE.

The focus of this mathematical practice is on analysis. Teachers encourage students to analyze by asking "why" questions. When students use properties, definitions, and contextual meanings to answer "why" questions, they are "using structure" to do and explain mathematics.

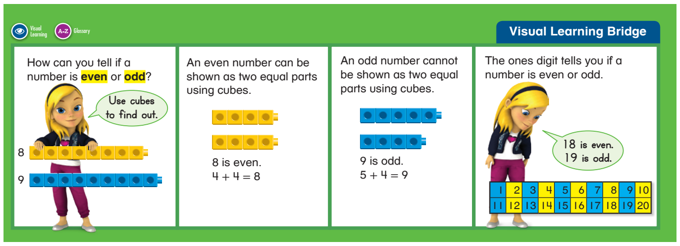
When students in Kindergarten and Grade 1 explain why an object in the real world is "like" a solid figure such as a cone, a cylinder, a sphere, or a cube, they are making use of structure as they identify and use the characteristics of each solid figure in order to find an object with similar characteristics.



*Source: **enVision**® Mathematics ©2020, Kindergarten, Lesson 12-5, p. 481 (Charles et al., 2020).*

Figure 12. Kindergarteners use attributes to connect real-world objects to geometric shapes.

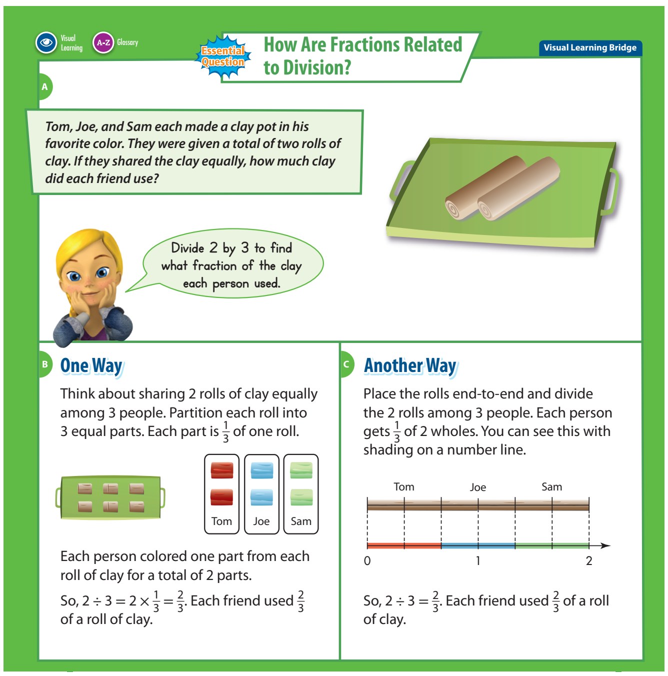
A student in Grade 2 or 3 might use the structure of even and odd numbers in order to explain that the sum of two even numbers is always even because it is broken into equal parts, and that the sum of an even number and an odd number is always odd because it cannot be broken into two equal parts. Students may extend using this structure to explain that the sum of two odd numbers is always an even number because each odd number has an extra 1, the two extra 1s form a pair, and therefore, the sum does NOT have an extra 1.



Source: ***enVision***® *Mathematics ©2020, Grade 2, Lesson 2-1, p. 62 (Charles et al., 2020).*

Figure 13. Grade 2 students use connecting cubes and number charts to explore the structure of even and odd numbers.

Students in Grades 4 and 5 use structure when they explain how fractions are related to division.



Source: ***enVision***® *Mathematics ©2020, Grade 5, Lesson 9-1, p. 386 (Charles et al., 2020).*

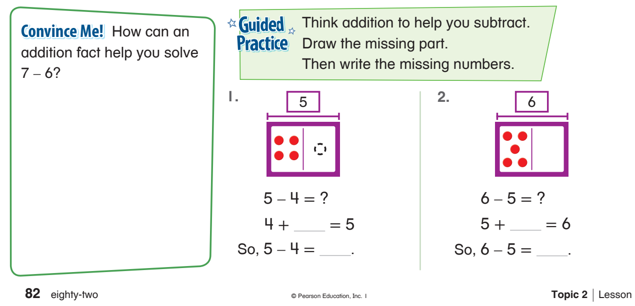
Figure 14. Grade 5 students use the meaning of division to explain how the fraction "two-thirds" is related to the quotient 2 divided by 3.

Across all grade levels, it is important for teachers to help students focus on the properties, definitions, and characteristics of the mathematics they are learning so that they can ask "why?" and use their mathematical knowledge to create an explanation.

MP8: LOOK FOR AND EXPRESS REGULARITY IN REPEATED REASONING.

Students who are asked to look for patterns and draw conclusions from those patterns are being encouraged to create generalizations. However, the patterns that students need to focus on are not just simple repetitions in sequences of shapes or numbers, but also repetitions in various procedures and their results.

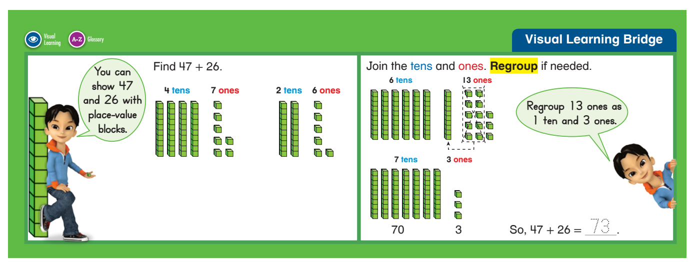
As students in Kindergarten and Grade 1 explore joining and separating situations, they draw conclusions about the relationship between addition and subtraction.



Source: ***enVision***® *Mathematics ©2020, Lesson 2-7, p. 82 (Charles et al., 2020).*

Figure 15. Grade 1 students use repeated reasoning to recognize the inverse relationship between addition and subtraction.

Students in Grades 2 and 3 use patterns in numbers to make generalizations about place value.



Source: ***enVision***® *Mathematics ©2020, Grade 2, Lesson 9-1, p.138 (Charles et al., 2020).*

Figure 16. Grade 2 students use place-value blocks and repeated reasoning to understand place value.

In Grade 4, students make generalizations about finding area by exploring questions such as, "How many one-inch square tiles fit into a 3 inch × 5 inch rectangle? Into a 3 inch × 6-inch rectangle? Into a 3 inch × 7-inch rectangle? What conclusion could you propose about the relationship between a rectangle's dimensions and its area?"

In Grade 5, students might be asked to find quotients and remainders for a set of related whole-number division problems in which the divisor is always 5 and the dividend increases by 1. They could answer the question, "What remainders occurred when you divided by 5?" Then students could find quotients and remainders for similar sets of division problems in which the divisor is 6, then 7, and then 8, and be asked to observe the remainders they get for each divisor. To encourage students to look for regularity, teachers would ask them to describe what they notice about each set of remainders related to its divisor. Based on this repeated reasoning, a teacher would then ask, "What conclusion about remainders could you propose based on these examples?" As students propose conclusions, teachers can then ask them to make use of structure by analyzing whether or not their generalizations are always true, why or why not they are always true, or under what conditions they are always true.

Summary

Students build mathematical thinking habits by engaging in mathematical practices. Rich mathematical tasks provide students opportunities to engage in the eight mathematical practices previously discussed. Students begin by making sense of the problem (MP1), using quantitative reasoning (MP2) and appropriate tools (MP5). Students use mathematics to model their understanding of the problem and design a process for its solution (MP4). Their approach to solving the problem may involve drawing a conclusion through repeated reasoning (MP8) as well as a recognition of patterns caused by the mathematical structure in the situation (MP7). As students persist in their work on the problem (MP1), they justify their reasoning and critique the reasoning of others (MP3), using precise language and notation (MP6). Although a single lesson seldom focuses on every mathematical practice, students experience each mathematical practice many times throughout the year. As students learn the process of asking themselves these questions while they are doing and learning mathematics, they internalize mathematical thinking habits.

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