SAVVAS

Research into Practice:

MATHEMATICS

Developing the “Full Package” of Procedural Fluency

BY DR. JENNIFER M. BAY-WILLIAMS

A Little Background About a Long Story...

If you have been in mathematics education for a while, you may recall classic discussions about number sense, calling it hard to define, easier to recognize and describe. In fact, we easily recognize that children who have number sense demonstrate this through their general understanding of number and operations, along with their ability and inclination to use this understanding in flexible ways to make mathematical judgments (McIntosh, Reys, Reys, Bana, and Farrell, 1997). We can describe number sense as the ability to decompose numbers naturally, to use the relationships among arithmetic operations to solve problems, to understand the base ten number system, to estimate, and to recognize the relative and absolute magnitude of numbers (NCTM, 2000).

As number sense became a featured part of reform initiatives, we tended to focus on developing *conceptual understanding*, while *procedural fluency* was largely overlooked, or even positioned as a set of skills that should be de-emphasized. Yet, developing mathematical proficiency requires a strong background in both, and in fact, a strength in one of these two areas can help to strengthen the other area.

To demonstrate, let’s revisit just one of the number sense descriptors: the ability to decompose numbers naturally. An example of this in Grades 1 and 2 is using the Make 10 strategy. A child recognizes that 7 + 4 can be solved by decomposing 4 into 3 + 1, then adding the 7 + 3 to “make 10”, and adding the 1 to equal 11. Visuals such as ten frames and number lines help students to see how they can decompose numbers (see Figure 1a and 1b). The number line is underutilized in early learning, but is critical to building an understanding of the relative size of numbers, and will help students generalize the Make 10 strategy for basic facts, to a Make 30 strategy for sums such as 28 + 67, and other problems that can be efficiently solved using a decomposition strategy.

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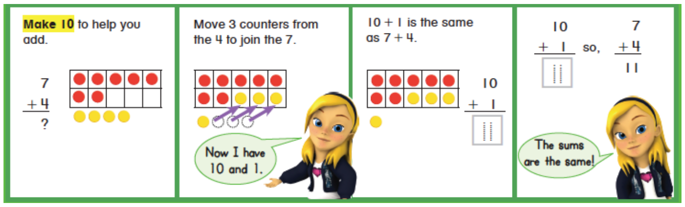
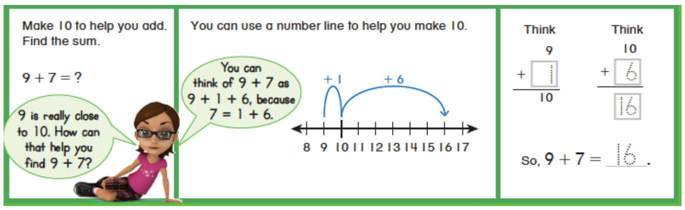


Source: **enVision®** Mathematics ©2020, *Grade 1, Topic 3, p. 126 (Charles et al., 2020).*

Figure 1. Ten frames and number lines help students learn and remember the Make 10 strategy.

1a. Ten frames and counters help students envision how to "make 10"



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1b. Number lines help students envision Make 10 and extend to other adding and subtracting strategies.

Decomposing can also be see in multiplication, as students can break numbers apart into tens and ones in order to multiply (see Figure 2). Thus decomposing, which requires conceptual understanding, is critical to procedural fluency.

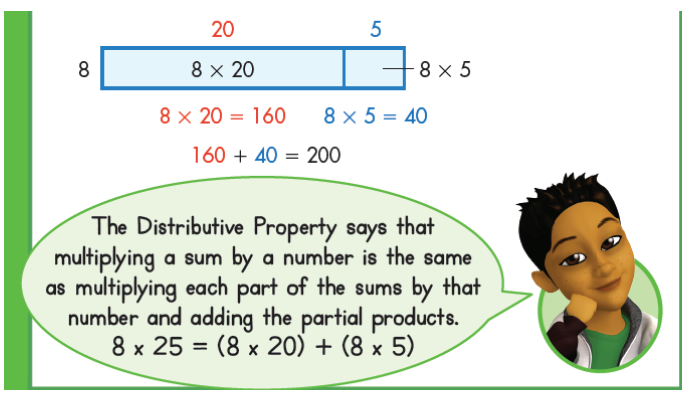

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Figure 2. An area model illustrates how one factor in 8 × 25 can be decomposed as an effective way to multiply.

THE FULL PACKAGE OF FLUENCY

Procedural fluency, like number sense, is best understood by describing what students who have it can do. Such students have "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (Kilpatrick, Swafford, and Findell 2001). Think of a topic in the K–5 curricula, such as adding within 1,000. With this topic in mind, ask yourself what it means for a student to be able to fluently add within 1,000. For too long and for too many, this has meant applying an algorithm correctly. Yet, this addresses only one of the four elements of procedural fluency. Not convinced? Let's look at an example:

$$299 + 436 =$$

Suppose student stacks these up, adds from the ones place to the hundreds place, regrouping all the way, and gets the correct answer. How would you rate that student on the four components of fluency?

Accurate:	Yes	No
Efficient:	Yes	No
Appropriate strategy selection:	Yes	No
Flexible:	Yes	No

The only certain Yes answer is "accurate." Applying the U.S. standard algorithm here is not especially efficient. Decomposing 436 into 1 + 435 and rethinking the sum as 300 + 435, for example, would be a more-efficient strategy. Therefore, the strategy selected was not appropriate, because all that regrouping was not necessary and was considerably slower. By not considering possible options for strategies, the student likely is not thinking flexibly about the operations (aha, number sense!).

Procedural skill has traditionally referred to accurate, smooth, rapid execution of mathematical procedures without regard to comprehension, flexibility, or strategy selection (Hiebert and Grouws 2007; Star 2005). Yet, recent state standards clearly intend to focus on the **full package of fluency** as something more than mastering the standard algorithm. The standards speak to the components of fluency across the grades, as this sampling of the standards indicates (bold added):

Add and subtract within 20. Add and subtract within 20, demonstrating **fluency** for addition and subtraction within 10. **Use strategies** such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Use place value understanding and properties of operations to add and subtract. Fluently add and subtract within 100 **using strategies** based on place value, properties of operations, and/or the relationship between addition and subtraction.

Understand properties of multiplication and the relationship between multiplication and division. Apply properties of operations as **strategies** to multiply and divide.

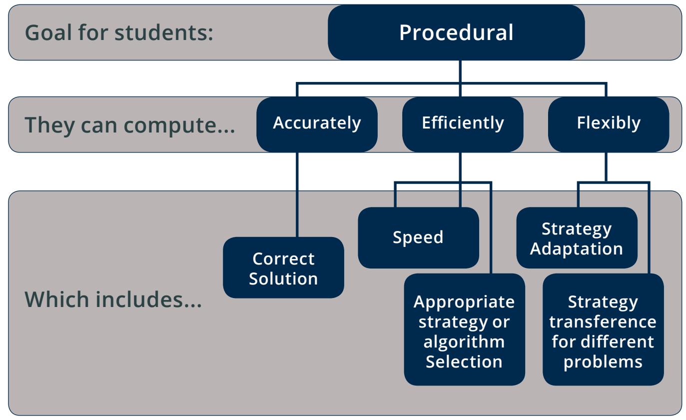
Multiply and divide within 100. Fluently multiply and divide within 100, **using strategies** such as the relationship between multiplication and division.

Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, **using strategies** based on place value and the properties of operations.

Perform operations with multi-digit whole numbers and with decimals to hundredths. Add, subtract, multiply, and divide decimals to hundredths, **using** concrete models or drawings and **strategies** based on place value, properties of operations, and/or the relationship between addition and subtraction; **relate the strategy to a written method** and explain the reasoning used.

Strategies, referenced repeatedly in the standards, are different from algorithms (Fuson and Beckmann 2012–2013). Strategies are "purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another" while algorithms are a "set of predefined steps for a class of problems" (NGA Center and CCSSO 2010, p. 85).

Based on a review of research on fluency, Bay-Williams and Stokes-Levine (2017) propose the flow chart in Figure 3 as a way to illustrate the full package of procedural fluency. This figure helps to illustrate the difference between superficial procedural knowledge, which may only include knowing one procedure with speed and accuracy, and deep procedural knowledge, which includes knowing strategies *and* algorithms, and knowing when to use them or adapt them. This deeper knowledge requires conceptual understanding.



Source: Bay-Williams, J. M., & Stokes Levine, A. (2017). The Role of Concepts and Procedures in Developing Fluency. In D. Spangler & J. Wanko (Eds.) ***Enhancing Professional Practice with Research Behind Principles to Actions.*** *Reston, VA: NCTM.*

Figure 3. Components of procedural fluency and their inter-relationships.

DEVELOPING FLUENCY IS A MARATHON, NOT A DASH

Developing procedural fluency (including connecting procedures to concepts) is a marathon. It requires use of visual representations and situations that enhance students' abilities to learn specific efficient strategies, and then learn to *select* the strategy that best fits a situation. Additionally, standard algorithms take time to develop. That time commitment, as well as misunderstood beliefs about the standard algorithms, can result in students abandoning their other strategies and using only the standard algorithm. Students do not "graduate" to the standard algorithm, never again to use the other strategies and algorithms they learned; rather the standard algorithm is *added* to a students' repertoire of strategies. Read this sentence again. Without this approach, students will not develop fluency, as described in Figure 3. When encountering a problem such as 299 + 349, the fluent child thinks, *"Which strategy should I use (from my repertoire) given the numbers in the problem?"* Going back to the four components of procedural fluency, this child has the potential for scoring a Yes in all four categories.

Developing procedural fluency is complex and takes time. It requires (1) knowing and understanding a variety of strategies and algorithms (fluency), (2) being able to judge one's repertoire to determine which strategy best fits the problem given (efficiency), and (3) being able to adapt a selected strategy to fit a problem (flexibility). These cognitive processes take time to develop (Baroody and Dowker 2003; LeFevre et al. 2006; Osana and Pitsolantis 2013; Schneider et al. 2011). Here we share five pragmatic, research-based suggestions for developing procedural fluency.

1. FOCUS ON WHY STRATEGIES OR ALGORITHMS WORK

The classic Bloom's (revised) taxonomy describes a hierarchy of thinking, from answering low-level knowledge questions to engaging in increasingly more complex levels of thinking (see Figure 4) (Fan & Bokhove, 2014). The simple act of asking "Why does that strategy work?" moves the focus of a mathematics lesson from knowledge (Level 1) to understanding (Level 2), from low-level to high-level. This is what standards mean when they include the phrasing "using properties" and "using place value": these truths about numbers provide the rationales for why particular strategies work. While students may not be able to name the properties of operations, they learn to use them to be flexible in how they solve problems.

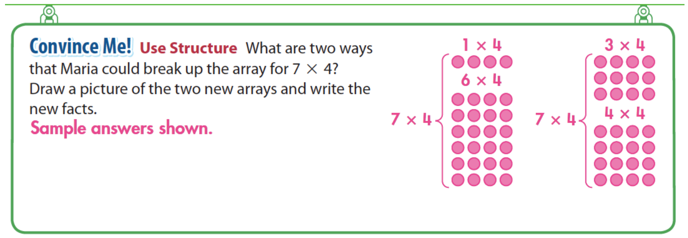
Higher-level Thinking Develops Procedural Fluency

Level of Thought (Fan of Bokhove, 2014)	Bloom's Level	Fluency with Algorithms
Level 3: Evaluation and Construction	Create Evaluate	• Comparing different algorithms • Judging efficiency of an algorithm • Constructing new algorithms (strategies) • Generalizing
Level 2: Understanding and Comprehension	Analyze Apply Understand	• Describing why a procedure works • Applying procedure in a complex situation
Level 1: Knowledge and Skills	Remember	• Telling the steps of a procedure • Carrying out steps in a straightforward situation

Source: Based on Fan, L. and Bokhove, C. (2014). Rethinking the Role of Algorithms in School Mathematics: A Conceptual Model with Focus on Cognitive Development." ZDM International Journals of Mathematics Education, 46(3), 481–492.

Figure 4. Bloom's (Revised) taxonomy as applied to procedural fluency with algorithms.

One way to help students see why a strategy works is by having students *generate how* they will apply a strategies and describe or illustrate their reasoning. The *Convince Me!* example in Figure 5, for example, provides an example of having students draw a picture (along with two possible ways counters might be used by a student).



Source: **enVision®** Mathematics ©2020, *Grade 3, Topic 3, p. 78 (Charles et al., 2020).*

Figure 5. Using break-apart (distributive property) as a strategy to learn basic multiplication facts.

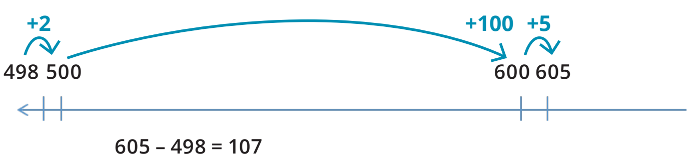
When students understand how and why they might break-apart a factor, they can then apply that idea to other numbers, including larger whole numbers, fractions, decimals, and so on.

2. FOCUS ON WHEN STRATEGIES OR ALGORITHMS WORK

Once students have learned strategies, and even while they are learning them, they must have significant opportunities to decide when to use them. For example, consider this example:

605 – 498 =

A student has several options. One student solves this by using a counting up strategy on an open number line:



After having the student explain *why* they selected this strategy, the question to pose to this student, and to other students, is:

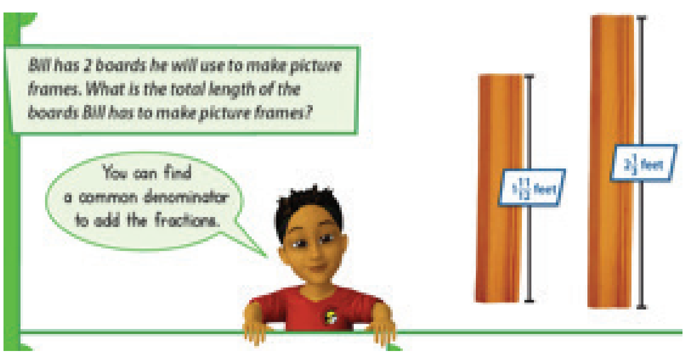
When is this a good strategy to use?

When is this strategy not going to work very well?

These *when* questions are necessary to help students generalize the strategy. Generalization is critical to both strategy selection and flexibility – two of the four components of fluency. Making good choices about strategies leads to making an *efficient* choice, and likely increases the chance of an accurate answer. In other words, *when* questions can support the full fluency package.

3. USE WORKED EXAMPLES

Worked examples (already solved problems) provide an opportunity to discuss both *why* a strategy works and *when* a strategy makes sense. Research has found that using **worked examples** is an effective instructional strategy to help students understand and solve problems (Renkl, 2014; Star & Verschaffel 2016). Worked examples provide an opportunity for students to verbalize their thinking through talking, writing, or drawing to describe the steps used to solve the problem, practices found to have a significant positive impact on students with difficulties in mathematics (Gersten and Clarke 2007). Worked examples also help students notice their own misconceptions, especially when tasks include self-explanation prompts (McGinn, Lange, & Booth 2015). Worked examples can showcase correct solutions (perhaps to highlight an interesting strategy), partially worked examples (perhaps to explore a different approach), or an incorrect strategy (perhaps to bring attention to a common error) (Van de Walle, Karp, & Bay-Williams, 2019). For example, in Figure 6, students are asked to consider if Bill's solution is correct. Bill's solution highlights an important concept of fraction addition: the addition of the parts (in this case twelfths).



Convince Me! Critique Reasoning Tom has 2 boards that are the same length as Bill's. He says that he found the total length of the boards by adding 28 twelfths and 23 twelfths. Does his method work? Explain.

Source: **enVision®** Mathematics ©2020, *Grade 4, Topic 7, p. 294 (Charles et al., 2020).*

Figure 6. Worked examples help students explore when and why a strategy might (or might not) work.

4. COMPARE PROCEDURES

Research on improving student achievement suggests that mathematics flexibility and conceptual understanding can be increased when students are asked to make comparisons (Fuson 2005; Rittle-Johnson and Star 2009; Rittle-Johnson, Star, and Durkin 2009, 2012; Star et al. 2015). Comparison tasks are cognitively higher level, or Level 3 in Bloom's (revised) taxonomy (Figure 4). A comparison task might ask students to compare problem *types*, or it might ask them to compare problem *solution methods* (Rittle-Johnson, Star, and Durkin 2009; Star et al. 2015). Practice sets can be strategically designed to have students look for patterns and make generalizations through comparing (Blanton 2008). Through such comparisons, students gain conceptual insights into the algorithms they are using and strengthen their number sense. To support the informal strategies discussed above, a set of multiplication problems might be created (see Figure 7).

1. 9 × 13=
2. 10 × 13=
3. 10 × 17=
4. 12 × 17=
5. 11 × 42=
6. 10 × 42=
7. 8 × 42=
8. Explain how #1 compares to #2.
9. Explain how #3 could help you solve #4.
10. Describe a general strategy you can use when multiplying a number by 9.

Figure 7. A problem set that lends to connecting procedures and concepts.

Sharing multiple solution strategies commonly takes place as you discuss solutions to a *Solve & Share* in ***enVision***® Mathematics (Charles et al., 2020), or at the end of lesson when discussing a selected problem. Importantly, this sharing should not be simply a show-and-tell, which often produces little higher-level thinking. The showcasing of strategies *sets up* the opportunity for strategy comparison, but the teacher must pose questions that focus students' attention on similarities and differences across solutions in order for students to reason at high levels about the strategies (NCTM, 2014).

5. MAKE CONNECTIONS EXPLICIT

The value of making connections explicit is one of the strongest findings in the research on classroom practices that support conceptual understanding and procedural fluency (Baker et al. 2014; Fuson 2005; Hiebert and Grouws 2007; Osana and Pitsolantis 2013). Making connections requires higher-level thinking, involving comprehension and even generalization (Levels 2 and 3 in Blooms [revised] taxonomy).

What are some ways to make these connections? First, using familiar and interesting situations provides relevance and prompts students to gauge whether a solution is reasonable (Osana and Pitsolantis 2013). Interesting, relevant situations can convert a rote exercise (Level 1) into an understanding or generalizing cognitive activity. Second, physical models illustrate the meaning behind mathematical symbols and operations (NCTM 2014; Osana and Pitsolantis 2013). It isn't enough to show illustrations one day, and symbols the next; students need opportunities to make connections between concrete visuals and the related symbols. Figure 8 illustrates how fraction strips are used to help students understand how to compare same numerator fractions, and then connect to the related symbolic representations.

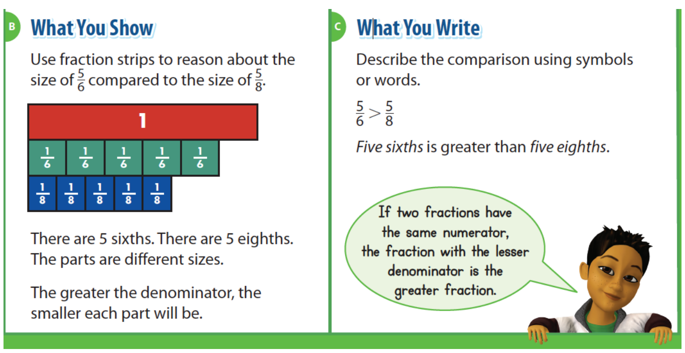


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Figure 8. Students need regular opportunities to connect concrete representations to abstract symbolic representations.

Attention to symbols, physical models, and the problem situations builds connections between concepts and procedures. Understanding these connections helps students know when and how to use strategies, and thereby leads to flexibility in solving problems.

Summary

The NCTM recommended Teaching Practice, *Build procedural fluency from conceptual understanding*, encompasses both procedures and concepts. Importantly, the student outcome for procedural fluency and conceptual understanding is higher-level cognition. The five research-based strategies described above must be integrated into daily mathematics teaching. The result will be the full package of fluency. A student having this full package is not just better prepared for some high-stakes assessment, but for all the mathematics that is to follow in later grades, and more importantly, for handling the mathematics of daily living.

References

Baker, Scott, Geva Esther, Michael J. Kieffer, Nonie Lesaux, Sylvia Linan-Thompson, Joan Morris, C. Patrick Proctor, Randi Russell, Russell Gersten, Joseph Dimino, Madhavi Jayanthi, Kelly Haymond, and Rebecca Newman-Gonchar. 2014. *Teaching Academic Content and Literacy to English Learners in Elementary and Middle School* (NCEE 2014-4012). Washington, DC: U.S. Department of Education.

Baroody, Arthur J. and Ann Dowker (Eds.). 2003. *The Development of Arithmetic Concepts and Skills: Constructing Adaptive Expertise.* Mahwah, NJ: Erlbaum.

Bay-Williams, J. M., & Stokes Levine, A. (2017). The Role of Concepts and Procedures in Developing Fluency. In D. Spangler & J. Wanko (Eds.) *Enhancing Professional Practice with Research Behind Principles to Actions.* Reston, VA: NCTM.

Blanton, Maria L. 2008. *Algebra in the Elementary Classroom: Transforming Thinking, Transforming Practice.* Portsmouth, NH: Heinemann.

Charles, Randall I., et al. 2020. ***enVision***® Mathematics ©2020. Glenview, IL: Pearson Education, Inc.

Fuson, Karen C., and Sybilla Beckmann. 2012–2013. "Standard Algorithms in the Common Core State Standards." In *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership:* 14–30.

Gersten, Russell and Benjamin S. Clarke. 2007. Effective Strategies for Teaching Students with Difficulties in Mathematics: NCTM Research Brief. Reston, VA: NCTM. **http://www.nctm.org/uploadedFiles/Research_and_Advocacy/research_brief_and_clips/Research_brief_02_-_Effective_Strategies.pdf**. (accessed July 28, 2016).

Hiebert, James, and Douglas A. Grouws. 2007. "The Effects of Classroom Mathematics Teaching on Students' Learning." In *Second Handbook of Research on Mathematics Teaching and Learning,* edited by F. K. Lester, 371–404. Charlotte, NC: Information Age.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. 2001. *Adding It Up: Helping Children Learn Mathematics,* edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell, Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press, 2001.

LeFevre, Jo-Anne, Brenda L. Smith-Chant, Lisa Fast, Sheri-Lynn Skwarchuk, Erin Sargla, Jesse S. Arnup, Marcie Penner-Wilger, Jeffrey Bisanz, and Deepthi Kamawar. 2006. "What Counts as Knowing? The Development of Conceptual and Procedural Knowledge of Counting from Kindergarten through Grade 2." *Journal of Experimental Child Psychology* 93: 285–303.

McGinn, Kelly M., Karin E. Lange, and Julie L. Booth. 2015. "A Worked Example for Creating Worked Examples." Mathematics Teaching in the Middle School 21: 26–33.

McIntosh, A., Reys, B., Reys, R., Bana, J., Farrell, B. (1997). Number sense in school mathematics: student performance in four countries. Perth, Australia: Mathematics, Science & Technology Education Centre, Edith Cowan University.

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA NCTM.

National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions.* Reston, VA: NCTM.

National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO]. 2010. *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Osana, Helen P. and Nicole Pitsolantis. 2013. "Addressing the Struggle to Link Form and Understanding in Fractions Instruction." *British Journal of Educational Psychology* 83: 29-56.

Renkl, Alexander. 2014. "Learning from Worked Examples: How to Prepare Students for Meaningful Problem Solving." In *Applying Science of Learning in Education: Infusing Psychological Science into the Curriculum, edited by* Victor Benassi, Catherine E. Overson, and Christopher M. Hakala, 118–130. **http://teachpsych.org/ebooks/asle2014/index.php. (accessed July 28, 2016)**.

Rittle-Johnson, Bethany, Jon R. Star, and Kelley Durkin. 2009. "The Importance of Prior Knowledge When Comparing Examples: Influences on Conceptual and Procedural Knowledge of Equation Solving." *Journal of Educational Psychology* 101: 836–852.

Rittle-Johnson, Bethany, and Jon R. Star. 2009. "Compared with What? The Effects of Different Comparisons on Conceptual Knowledge and Procedural Flexibility for Equation Solving." *Journal of Educational Psychology* 101: 529–544.

Schneider, Michael, Bethany Rittle-Johnson, and Jon R. Star. 2011. "Relations among Conceptual Knowledge, Procedural Knowledge, and Procedural Flexibility in Two Samples Differing in Prior Knowledge." *Developmental Psychology* 47: 1525–1538.

Star, Jon. R. 2005. "Reconceptualizing Procedural Knowledge." *Journal for Research in Mathematics Education* 36: 404–411.

Star, Jon R., Anne Foegan, Matthew R. Larson, William G. McCallum, Jane Porath, Rose Mary Zbiek, Pia Caronongan, Joshua Furgeson, Betsy Keating, and Julia Lyskawa. 2015. *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students* (NCEE 2015-4010). Washington, DC: U.S. Department of Education.

Star, Jon R. and Lieven Verschaffel. 2016. Providing Support for Student Sense Making: Recommendations from Cognitive Science for the Teaching of Mathematics. In *Compendium for Research in Mathematics Education,* edited by Jinfa Cai. Reston, VA: National Council of Teachers of Mathematics.

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