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Research into Practice:

MATHEMATICS

Developing Conceptual Understanding: The Case of Place Value

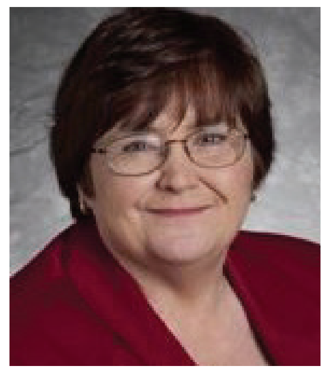
BY DR. JANET H. CALDWELL

Over time, it seems that the major focus for teaching elementary mathematics has shifted back and forth between procedural skills and conceptual understanding, with problem solving also appearing as an emphasis. These are not competing aspects of mathematical proficiency, however, but different aspects of mathematics that develop in an iterative manner, supporting each other, throughout the elementary years (and beyond).

What Is a Concept?

It may be helpful to think about a mathematics concept as a big idea or an essential understanding. Sometimes we describe a concept by using a simple noun, such as “place value,” but a big idea typically describes a network of essential understandings relating to it. For place value, for example, some of the big ideas might include the following:

- A group of 10 can be thought of as a single object or a set of 10 separate objects.
- By putting together sets of 10 and breaking apart sets of 10, we can find equivalent representations of any number.
- The position of a digit in a number tells its value.
- It takes 10 of a number in one place-value position to make a number in the next greatest place-value position. For example, it takes 10 tens to make 1 hundred.
- A digit in one place represents 10 times what it would represent in the place to its right and one-tenth of what it would represent in the place to its left.



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The relationship between a concept such as place value and a related skill such as adding two-digit whole numbers is complex. For example, to add 47 + 35, a student may use one of several strategies. He might use his place-value understanding to combine ones from each addend to get 10 ones, as shown in the top example in Figure 1. Alternatively, he might add the tens from each addend and then add the ones from each addend, as shown in the bottom left example in Figure 1. Or, he may add the ones, regroup to form a 10, and then add the tens (the United States standard algorithm), as shown in the bottom right example in Figure 1. Knowing the steps to follow in a particular algorithm is a skill, but knowing why the steps make sense requires understanding place vale (as well as addition).

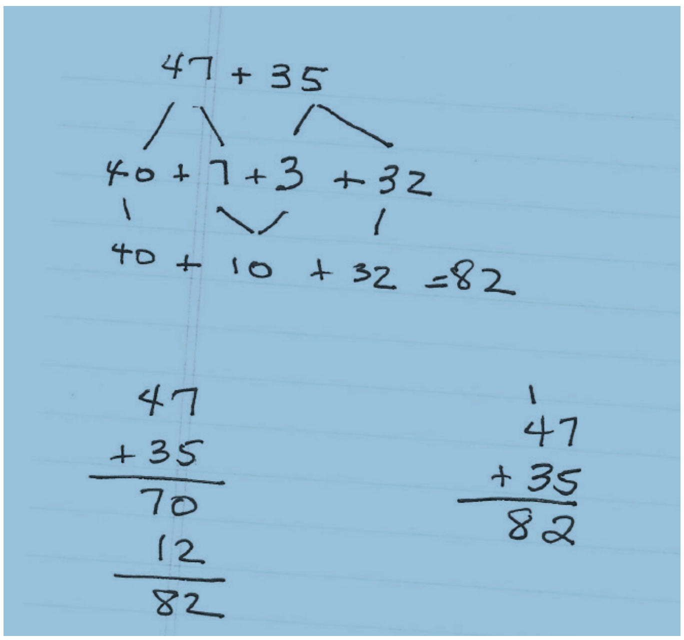


Figure 1. Three different ways that a student might solve 47 + 35 using understanding of place value and addition.

What Is Conceptual Understanding?

In *Adding It Up*, conceptual understanding is characterized as “an integrated and functional grasp of mathematical ideas” (National Research Council 2001, 118). It is more than knowing facts and computational procedures; it involves knowing why a mathematical idea is important and when it is useful. Conceptual understanding develops over time as students organize their knowledge into a coherent, integrated mental structure. This understanding is most useful when the structure relates mathematical concepts to methods and computational procedures. Conceptual understanding includes being able to represent a variety of situations in different ways, to see the connections between those representations, and to know how to use each representation for different purposes.

Why Focus on Conceptual Understanding?

Learning how to perform computational algorithms is not sufficient preparation for life in today’s technological world. Adults carry many devices that can perform most calculations, from cell phones to tablets and computers. Students must know more in today’s world; they must understand what they are doing and why they are doing it; they must select appropriate ways of solving problems and doing computations. Hiebert (2003, 53) summarizes much of what we know today by saying, “Understanding is the key to remembering what is learned and being able to use it flexibly.”

TEACH FOR UNDERSTANDING

Teaching for understanding has several important outcomes.

- When students understand what they are doing, they remember it better. For example, my son learned how to multiply two-digit numbers by rote in third grade, but he forgot how to do so by the beginning of fourth grade. When I used partial products to help him understand what he was doing, he understood and remembered the procedure.
- Students make more correct computations when they understand the mathematical concepts related to their work.
- Understanding is essential to solving problems. If students have a rich and deep understanding of a concept, then they can more easily apply that concept in a problem-solving situation.
- When students understand concepts, they link ideas together in meaningful ways so that there are fewer discrete things to learn. Mathematics is not about memorizing how to get right answers; it is about recognizing and using underlying structures and patterns to see how seemingly disparate ideas are connected.
- Students who understand concepts are also able to apply mathematical practices, such as constructing viable arguments and justifying their thinking.

As teachers, we believe that understanding is an important goal for our students, but sometimes we settle for students being able to perform computations. This emphasis on carrying out procedures shortchanges our students, making it more difficult for them to learn new concepts and effectively apply what they have learned in new situations.

CONCEPTS DEVELOP OVER TIME

Students do not develop understanding of a concept in a linear fashion. In fact, conceptual understanding develops in a variety of ways (Lesh and Yoon 2004). Some researchers (Thompson et al. 2014) describe conceptual understanding as a cloud with parallel developments in interacting branches of mathematics. For example, understanding of place value is connected to and affected by understanding of the metric system.

Just as when learning a new language, students often understand more than they are able to verbalize. A first grader, for example, may understand that 13 represents 13 ones or one group of 10 and 3 more ones. When asked to explain how she knows this, however, she may reply, “I just know it” and be unable to justify her thinking.

Conceptual understanding of place value is neither developed while teaching a single topic nor during a single year of instruction, but rather begins with relatively naïve understandings in kindergarten and grows throughout the elementary years and even beyond. The table below shows one version of the essential understandings related to place value (Figure 2). Students in kindergarten and first grade work with numbers up to 100, while students in Grades 2 and 3 work with numbers to 1000 and fourth graders work with greater whole numbers. In Grade 5, students work with decimal place value. Each year provides an opportunity for students to expand, enrich, and apply their understanding of place value.

Grade	Standards
K	Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.
1	Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: a. 10 can be thought of as a bundle of ten ones—called a “ten.” b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones). d. Decompose two-digit numbers in multiple ways (e.g., 64 can be decomposed into 6 tens and 4 ones or into 5 tens and 14 ones).
2	Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens — called a “hundred.” b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
	Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
3	Use place-value understanding to round whole numbers to the nearest 10 or 100.
	Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
	Multiply one-digit whole numbers by multiples of 10 in the range 10–90 using strategies based on place value and properties of operations.
4	Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.
	Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form.
5	Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
	Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000).

Source: National Governors Association Center for Best Practices, Council of Chief State School Officers. 2010. Common Core State Standards for Mathematics.

Figure 2. Standards showing how conceptual understanding of place value develops each year from kindergarten through Grade 5.

Researchers have broken down understanding of place value even further. A group of Australian researchers describe three levels of understanding of place value for two-digit numbers (Wright et al. 2006).

Level 1: Students understand 10 as 10 ones.

Level 2: Students see 10 as a single unit.

Level 3: Students easily work with units of 10.

Others have looked at student responses in individual interviews where students are asked about a two-digit number, such as 47 (adapted from Van de Walle, Karp, and Bay-Williams 2019, 227). In the Digit-Correspondence Task, students are first asked to count a group of 47 blocks. Then the student is asked to write the number that tells how many there are. Next, each digit (starting with the ones place) is circled and the student is asked, "Does this part of your 47 have anything to do with how many blocks there are?" Based on responses to the task, Ross (1989) identified five distinct levels of understanding of place value.

Level 1: ***Single numeral.*** The individual digits 4 and 7 have no meaning to the student; he sees 47 individual blocks.

Level 2: ***Position names.*** The student can identify each place-value position in the numeral but is unable to connect these single digits to the blocks.

Level 3: ***Face value.*** The student matches each digit with that number of blocks. For example, the 4 is matched with 4 blocks.

Level 4: ***Transition to place value.*** The student explains that the 4 represents 40 blocks but is unable to think of these as 4 groups of 10.

Level 5: ***Full understanding.*** The student is able to match the 4 in 47 with 40 blocks as well as with 4 groups of 10.

What Instructional Strategies Can Teachers Use to Help Develop Conceptual Understanding?

Recognizing that conceptual understanding is a vital part of improving student learning of mathematics is only the first step. Teachers must use instructional strategies that help students develop their conceptual understanding on an everyday basis. Five specific strategies are important to accomplish this: (1) building skills and procedures from concepts, (2) using problem-based teaching and learning, (3) using multiple representations, (4) focusing on reasoning and communication, and (5) helping students make connections between concepts.

1. BUILDING SKILLS AND PROCEDURES FROM CONCEPTS

This strategy is based on the *Principles to Actions* Effective Teaching Practice: "Build procedural fluency from conceptual understanding" (NCTM 2014, 10). The idea of building 'from' does not mean to teach concepts first, and then teach procedures; it does mean that students need a conceptual understanding in order to become fluent with skills and procedures. In terms of curriculum, this means that a concept such as adding two-digit numbers must have a daily focus on the skills and on understanding. An effective mathematics lesson, then, includes both an essential understanding and a skill objective, so that students are able to develop their mental network in both ways. The example below shows the skill objective and essential understanding for Grade 4, Lesson 1-1, in **enVision®** Mathematics ©2020 (Figure 3).

Lesson	Objective	Essential Understanding
1-1 **Numbers Through One Millon**	Read and write numbers through one million in expanded form, with numerals, and using number names.	Our number system is based on groups of ten. Whenever we get 10 in one place value, we move to the next greater place value.

Source: ***enVision***® *Mathematics ©2020, Grade 4 TE p. 1A*

Figure 3. The skill objective and essential understanding for a Grade 4 lesson.

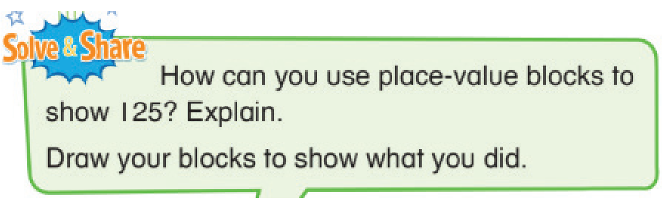
To help students develop conceptual understanding, it is important for teachers to focus on specific concepts and orchestrate classroom discussions to allow students to share and discuss their understandings.

2. USING PROBLEM-BASED TEACHING AND LEARNING

In problem-based teaching and learning, students are introduced to new content through a problem-solving task. The lesson begins with the teacher posing a problem and discussing it briefly with the class. Then the students work on the problem, usually in pairs or a small group, with the teacher facilitating and monitoring student work while listening to how students are thinking about strategies and modeling the problem. The teacher then selects student work to be shared in a whole class discussion, where he makes the mathematics to be learned more explicit. Lambdin (2003, 11) describes problem-based instruction, saying, “Teachers can help and guide their students, but understanding occurs as a by-product of solving problems and reflecting on the thinking that went into those problem solutions.”

In problem-based teaching and learning, cognitive demand is raised to a higher level, where students must struggle to understand and make connections between prior knowledge and the new situation, building their mental networks. To accomplish this, problem tasks must be of high quality, and teachers must support this productive struggle (Charles 2018).

Consider the following example from Grade 2, Lesson 9-2 (Figure 4).



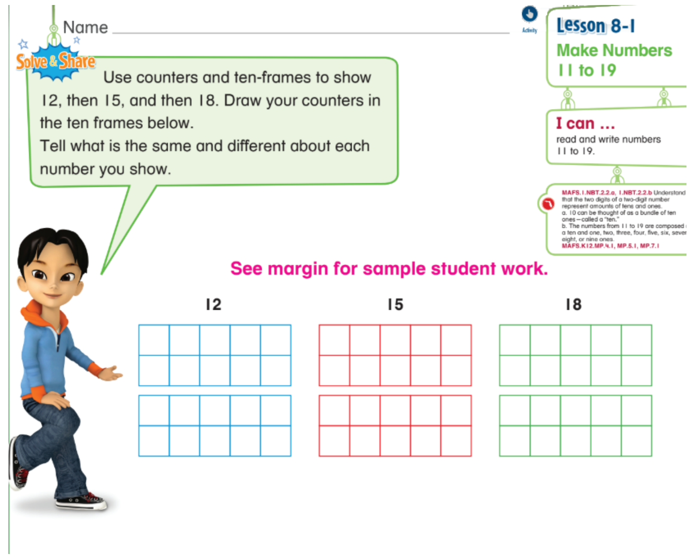
*Source: **enVision**® Mathematics ©2020, Grade 2 SE p 381*

Figure 4. Grade 2 place-value problem.

Teachers who use a problem-based teaching and learning approach with challenging, high-quality problems support students in productive struggle, while maintaining a focus on conceptual understanding, which leads to improved conceptual understanding and student achievement.

3. USING MULTIPLE REPRESENTATIONS

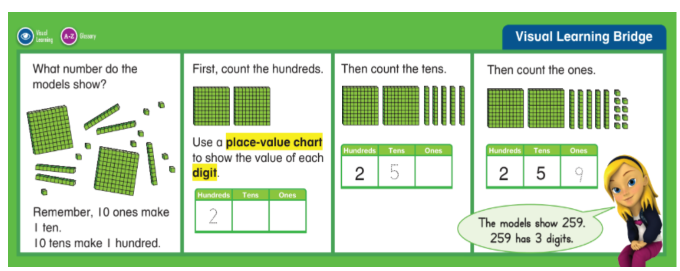
Another instructional strategy useful in improving student understanding involves students using multiple representations. Instruction on a new topic often begins with students using concrete materials, such as ten-frames, straws or sticks, or base-ten blocks. In the following place-value task from Grade 1, Lesson 8-1, students use counters to make groups of 10 (Figure 5).



*Source: **enVision**® Mathematics ©2020, Grade 1 SE p.325.*

Figure 5. Grade 1 place-value task in which students represent three teen numbers with counters and ten-frames.

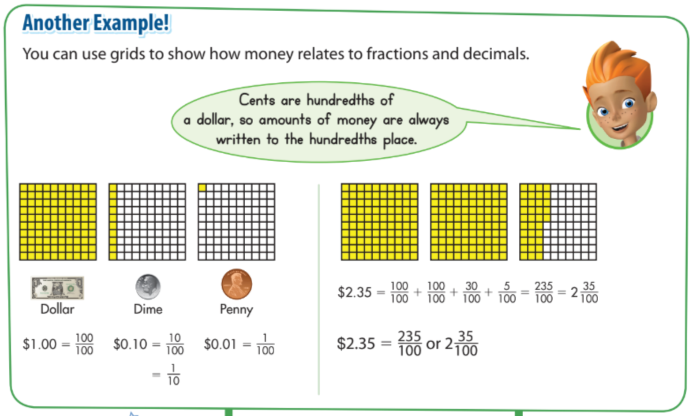
For place value, it is important to begin instruction in the early grades by using materials that students can use to construct their own groups of 10, such as counters, interlocking cubes, straws, or sticks. As students begin to understand how 10 can be thought of as either 10 separate objects or a single group of 10, they can move to using grouped concrete materials, such as base-ten blocks, as shown in the following example from Grade 2, Lesson 9-2 (Figure 6).



*Source: **enVision**® Mathematics ©2020, Grade 2 SE p. 382.*

Figure 6. Grade 2 use of place-value blocks to represent numbers.

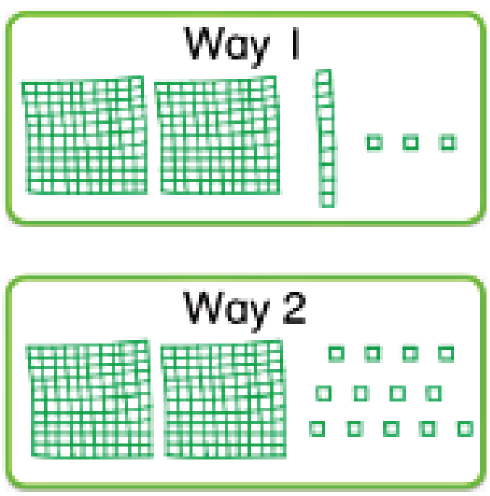
When students use place-value blocks, they are able to see the relationship between any two place values; they can see that 10 unit cubes (ones) make up 1 long (ten) and 10 longs (tens) make 1 flat (hundred). Models such as chips or money do not have this property but may be useful for older students, as shown in the following example from Grade 4, Lesson 12-1 (Figure 7).



Source: **enVision**® Mathematics ©2020, *Grade 4 SE p. 447.*

Figure 7. Grade 4 use of money to illustr ate decimals.

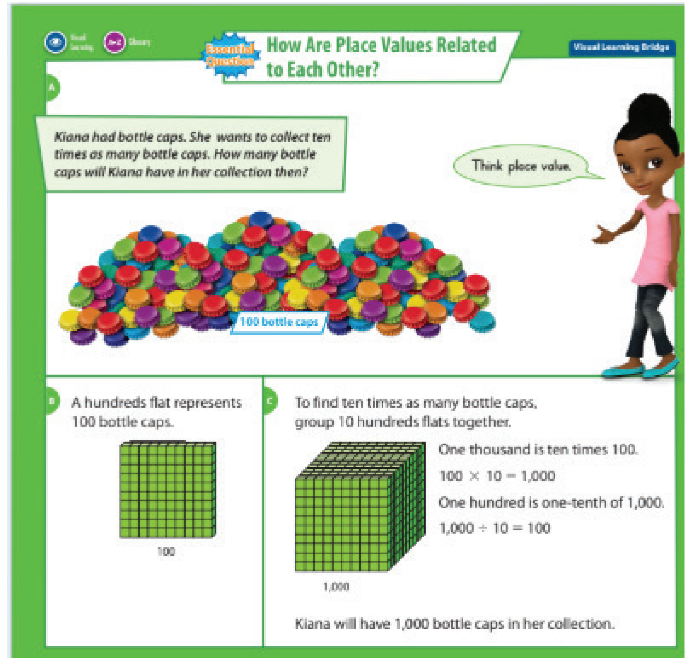
While instruction begins with the use of concrete representations, it should progress to pictorial visual representations and to symbolic representations. Teachers can help students explain their work by using pictures and symbols as students explain how they solved a problem, as shown in the following example from Grade 2, Lesson 9-5 (Figure 8).



Source: **enVision**® Mathematics ©2020, *Grade 2 TE p. 393.*

Figure 8. Grade 2 drawings of how a student used place-value blocks to show 213 in two different ways.

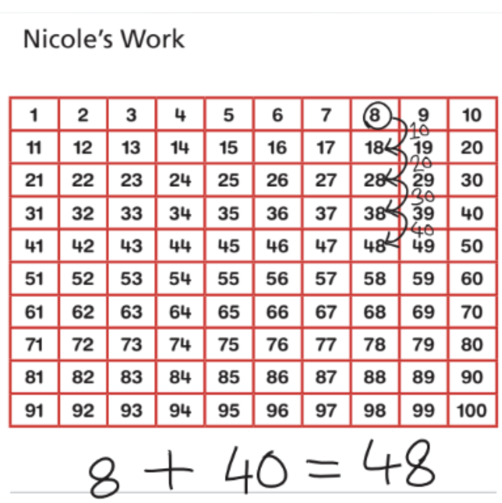
Because students today are often more familiar with visual representations, it can be helpful to use animation and visual displays to make mathematical concepts more visual (Murphy 2020). The following example from Grade 4, Lesson 1-2, shows this kind of visual representation (Figure 9).



Source: **enVision**® Mathematics ©2020, *Grade 4 SE p 10.*

Figure 9. Grade 4 use of visual displays that show how place values are related to each other.

A hundred chart can also be a helpful representation for students as they develop their understanding of place value. The example below from Grade 1, Lesson 10-3, shows how a student solved a problem using a hundred chart (Figure 10).



Source: **enVision**® Mathematics ©2020, *Grade 1 TE p 409).*

Figure 10. Grade 1 student solution using hundred chart.

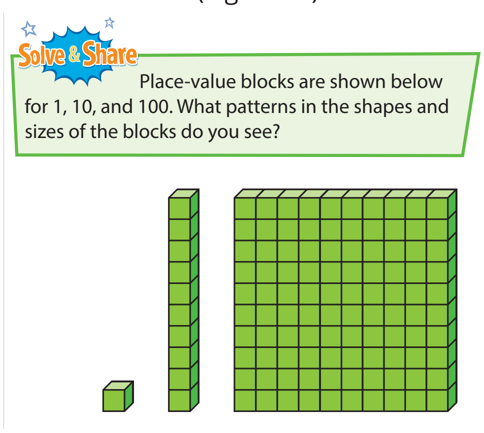
Care must be taken when using multiple representations to ensure that students focus on concepts rather than rote learning. For example, a typical place-value question in a text might be the following:

345 = ___ hundreds + ___ tens + ___ ones

Students can fill in these blanks without any thought about place-value concepts; they recognize that the first digit goes in the first blank, the second in the next blank, and the last in the last blank. This is why multiple representations are important. For example, students can represent this value with blocks or drawings of blocks. You can also ask students to give more than one answer. In this case, students might record the answer as 3 hundreds + 4 tens + 5 ones as the obvious answer, but include other equivalent answers such as 2 hundreds + 14 tens + 5 ones, and so on. They may use tools to generate these expressions, or they may use mental math. Linking multiple representations and connecting them in a mental network strengthens and deepens student understanding.

4. FOCUSING ON REASONING AND COMMUNICATION

Ideally, all students should be able to fully explain concepts using combinations of words, pictures, diagrams, and numbers. As teachers, however, we all recognize that this important goal can only be met by emphasizing math talk continually. Students need daily opportunities to express their understandings, whether to another student or to the class as a whole. Teachers need to maintain a high level of rigor in asking questions that require students to think deeply about mathematics rather than just asking students to answer a simple routine question. An example of a more rigorous type of question appears in Grade 4, Lesson 1-2, and is shown below (Figure 11).



Source: **enVision®** Mathematics ©2020, *Grade 4 SE p. 9.*

Figure 11. Grade 4 question that requires students to reason and communicate about place value.

Students need to have opportunities to construct mathematical arguments and justify their own thinking, but they also need to learn to listen to arguments presented by other students and constructively critique these arguments. The following example from Grade 4, Lesson 1-5, requires students to critique a given math argument and construct their own math arguments (Figure 12).

Solve & Share

The land areas of three states are shown in the table. Mickey said Alaska's land area is about 10 times greater than Georgia. Explain why Mickey is or is not correct. Construct a math argument to support your answer.

DATA

State	Land Area (in square miles)
Alaska	570,641
Georgia	57,513
Hawaii	6,423

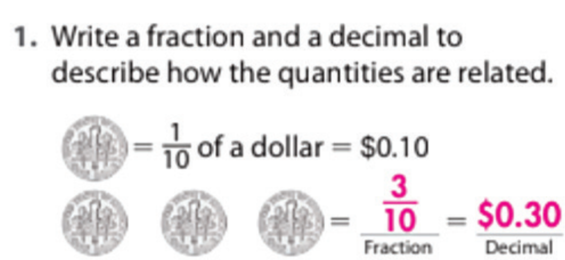
Source: **enVision®** Mathematics ©2020, *Grade 4 SE p 21.*

Figure 12. Grade 4 problem in which students critique a math argument and construct their own math arguments.

In order to develop rich and deep understanding of mathematical concepts, teachers must provide daily opportunities for students to communicate and reason about mathematics, but we must also listen carefully to students, to question their assumptions, and to provide feedback to them.

5. HELPING STUDENTS MAKE CONNECTIONS BETWEEN CONCEPTS

As students develop their understanding of a mathematical concept, they are learning new ideas and building links between those new ideas and existing ones. Teachers can help students build stronger connections by asking about relationships between ideas and by explicitly pointing them out. For example, when students are learning about metric lengths, they should connect that work to what they already know about place value. Other connections to place value include money and powers of 10, as shown in the following example from Grade 4, Lesson 12-5 (Figure 13).



Source: **enVision®** Mathematics ©2020, *Grade 4 SE p. 463.*

Figure 13. Grade 4 problem that connects decimal place value to money.

Some connections may seem obvious to adults but often are not apparent to students. For example, when students extend their understanding of place value from two-digit numbers to three-digit numbers, then to greater numbers, and finally to decimals, they need to recognize all of the connections inherent in this mathematical structure. They need to note and discuss how working with two-digit numbers is like working with three-digit numbers; they need to look for and express regularity in repeated reasoning by making generalizations about place value.

Conclusion

Conceptual understanding is one of the core elements of mathematical proficiency. Learning mathematics without understanding is like trying to follow directions to build a bridge without ever having seen one. Lambdin (2003, 7) puts it this way:

> "Nothing is more rewarding than the confident feeling that ideas make sense; and nothing is more frustrating than not understanding. Students who do not understand an idea often feel so discouraged and defeated that they give up even trying to learn... By contrast, to understand something is a very motivating and intellectually satisfying feeling. When ideas make sense to students, they are prompted to learn by their desire for even deeper understanding. They want to learn more because feeling successful in connecting new ideas with old is an exhilarating experience."

This paper has illustrated how place-value concepts are critical to developing both conceptual understanding and procedural skill with operations throughout K–5. The five instructional strategies described extend to all topics. When they are implemented on a daily basis, students will understand mathematics, and therefore become competent and confident mathematicians.

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0220.SRM.DI.Mat581M914

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